

Gieseke-Petri divisors and Brill-Noether theory of K3-sections

DISSERTATION

zur Erlangung des akademischen Grades

Dr. Rer. Nat.
im Fach Mathematik

eingereicht an der
Mathematisch-Wissenschaftlichen Fakultät II
Humboldt-Universität zu Berlin

von
Margherita Lelli-Chiesa

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Jan-Hendrik Olbertz

Dekan der Mathematisch-Wissenschaftlichen Fakultät II:
Prof. Dr. Elmar Kulke

Gutachter:

1. Prof. Dr. Marian Aprodu
2. Prof. Dr. Gavril Farkas
3. Prof. Dr. Daniel Huybrechts

eingereicht am: 1 Juni 2012

Tag der mündlichen Prüfung: 20 September 2012

Acknowledgements

First of all, I want to thank my advisor Gavril Farkas for introducing me to the wonders and surprises of research in Algebraic Geometry. Thanks to him, I got in touch with experienced and brilliant researchers since the very first months of my Ph.D. programme, I attended compelling conferences and I learnt about a wide variety of fascinating topics at the cutting edge of mathematics. I discovered very soon how stimulating, attractive and widespread the scientific community is. Also thank you for trusting me and letting me free of going to Rome quite often!

During these years, I have benefitted from inspiring conversations with several mathematicians, especially Peter Newstead and Marian Aprodu. I am grateful to Peter Newstead, among other things, for giving me the opportunity of spending a nice and productive period in Cambridge and for his availability. A special thanks goes to Enrico Arbarello and Riccardo Salvati Manni, who first taught me Algebraic Geometry during my bachelor and master programmes in Rome and kept supporting and encouraging me also during my graduate studies. Enrico has always shown a big enthusiasm for the topics I have been working on. Riccardo is constantly helpful, affective and available and has given me important advice, countless many times.

I will remember the years spent in Berlin as a happy and lively period. Both the algebraic and the arithmetic groups are friendly and harmonious and I consider most of my colleagues (I will not mention all of them) close friends. I especially thank Michele Bolognesi, Remke Kloosterman, Hartwig Mayer, Cristina Manolache and my “big brother” Nicola Tarasca, who made me feel at home since soon after my arrival in Berlin, helped me overcome the scientific frustrations experienced at the beginning of the Ph.D., both answering a lot of mathematical questions and providing me with enjoyable distractions. All of them are trustworthy confidants. Also thank you for the dangerous transport of a very heavy washing machine! I thank my “small brother” Fabian Müller for having corrected the (many) German mistakes in the abstract of this thesis. Berlin also offered me the nice environment at the BMS, to which I express my gratitude for supporting me financially.

Alessandro, thank you for the many English language corrections, for helping me with some computations and explaining me that calculating is not the hardest part of the job (you didn’t convince me), for the many pizza at “I due forni”, for listening to my grumbling about mathematics and for the encouragement and psychological support since the very beginning of the Ph.D. programme (in tempi ancora non sospetti) until now!

Last but not least, I would like to thank all my family, who was always present, despite being in another country. They have visited me so often in Berlin that they are now saying they will miss this city, and they have heard me complaining about problems,

counterexamples, Ext groups and sheaves so much that they finally got familiar with such words, even without having ever learned mathematics!

Abstract

We investigate Brill-Noether theory of algebraic curves, with special emphasis on curves lying on $K3$ surfaces and Del Pezzo surfaces.

In Chapter 2, we study the Gieseker-Petri locus GP_g inside the moduli space M_g of smooth, irreducible curves of genus g . This consists, by definition, of curves $[C] \in M_g$ such that for some r, d the Brill-Noether variety $G_d^r(C)$, which parametrizes linear series of type g_d^r on C , either is singular or has some components exceeding the expected dimension. The Gieseker-Petri Theorem implies that GP_g has codimension at least 1 in M_g and it has been conjectured that it has pure codimension 1. We prove this conjecture up to genus 13; this is possible since, when the genus is low enough, one is able to determine the irreducible components of GP_g and to study their codimension by “ad hoc” arguments and methods of classical Brill-Noether theory.

Lazarsfeld’s proof of the Gieseker-Petri-Theorem by specialization to curves lying on general $K3$ surfaces suggests the importance of the Brill-Noether theory of $K3$ -sections for a better understanding of the Gieseker-Petri locus. Linear series on curves lying on a $K3$ surface are deeply related to the so-called Lazarsfeld-Mukai bundles. In Chapter 3, we study the stability of rank-3 Lazarsfeld-Mukai bundles on a $K3$ surface S , and show it encodes much information about nets of type g_d^2 on curves C contained in S . When d is large enough and C is general in its linear system, we obtain a dimensional statement for the variety $G_d^2(C)$. If the Brill-Noether number is negative, we prove that any g_d^2 is contained in a linear series which is induced from a line bundle on S , as conjectured by Donagi and Morrison. Some applications towards higher rank Brill-Noether theory and transversality of Brill-Noether loci are then discussed.

Chapter 4 concerns syzygies of any given curve C lying on a Del Pezzo surface S . In particular, under some mild hypotheses on the line bundle $\mathcal{O}_S(C)$, we prove that C satisfies Green’s Conjecture, which implies that the existence of some special linear series on C can be read off the equations of its canonical embedding. A result of Aprodu reduces Green’s Conjecture to a dimensional condition for some Brill-Noether varieties, that we verify using vector bundle techniques.

Zusammenfassung

Diese Dissertation untersucht Brill-Noether-Theorie der algebraischen Kurven, unter besonderer Berücksichtigung von Kurven auf K3-Flächen und Del-Pezzo-Flächen.

In Kapitel 2 studieren wir den Gieseker-Petri-Ort GP_g im Modulraum M_g der glatten irreduziblen Kurven vom Geschlecht g . Dieser Ort wird definiert durch Kurven mit einer Brill-Noether-Varietät $G_d^r(C)$, die singular ist oder deren Dimension größer als erwartet ist. Der Satz von Gieseker-Petri impliziert, dass GP_g mindestens Kodimension 1 hat, und es wurde vermutet, dass er von reiner Kodimension 1 ist. Wir beweisen diese Vermutung für Geschlecht höchstens 13. Dies wird dadurch ermöglicht, dass man für kleine Geschlechter die Dimension der irreduziblen Komponenten von GP_g mittels "ad hoc"-Beweisführungen und Methoden aus der klassischen Brill-Noether-Theorie untersuchen kann.

Lazarsfelds Beweis des Gieseker-Petri-Theorems mittels Kurven auf allgemeinen K3-Flächen deutet darauf hin, dass die Brill-Noether-Theorie von K3-Schnitten wichtig ist, um den Gieseker-Petri-Ort besser zu verstehen. Linearscharen von Kurven, die auf K3-Flächen liegen, stehen in tiefgehender Beziehung zu sogenannten Lazarsfeld-Mukai-Vektorbündeln. In Kapitel 3 untersuchen wir die Stabilität der Lazarsfeld-Mukai-Vektorbündel vom Rang 3 auf einer K3-Fläche S , und wir zeigen, dass sie viele Informationen über Netze vom Typ g_d^2 auf Kurven in S enthalten. Wenn d groß genug ist, erhalten wir eine obere Schranke für die Dimension der Varietät $G_d^2(C)$. Wenn die Brill-Noether-Zahl negativ ist, beweisen wir, dass jedes g_d^2 in einer von einem Geradenbündel induzierten Linearschar enthalten ist, wie von Donagi und Morrison vermutet wurde. Schließlich werden Anwendungen in der Brill-Noether-Theorie der Vektorbündel von höherem Rang und der Transversalität von Brill-Noether-Orten diskutiert.

Kapitel 4 befasst sich mit Syzygien einer gegebenen Kurve C , die auf einer Del-Pezzo-Fläche liegt. Unter milden Annahme über $\mathcal{O}_S(C)$ beweisen wir insbesondere, dass C die Greens Vermutung erfüllt, die impliziert, dass die Existenz gewisser spezieller Linearscharen auf C von den Gleichungen ihrer kanonischen Einbettung abgelesen werden kann. Ein Ergebnis von Aprodu führt Greens Vermutung auf eine Dimensionsbedingung für gewisse Brill-Noether-Varietäten zurück, die wir mittels Vektorbündelmethode verifizieren.

Contents

Acknowledgements	iii
Abstract	v
Zusammenfassung	vii
1 Introduction	1
1.1 The Brill-Noether Theorem	2
1.2 The Gieseker-Petri Theorem	4
1.3 M_g is of general type for $g \geq 24$	6
1.4 Lazarsfeld's proof of the Gieseker-Petri Theorem	7
1.5 Propagation of linear series on $K3$ -sections	9
1.6 Gonality and Clifford dimension of $K3$ -sections	12
1.7 Green's Conjecture for curves on $K3$ surfaces	14
1.8 Moduli spaces of sheaves on $K3$ surfaces	17
1.9 Outline of the results	19
Bibliography	27
2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$	29
2.1 Introduction	29
2.2 Divisorial components of GP_g	31
2.3 Some useful inclusions	34
2.4 Proof of Theorem 2.1.1 in genus 9, 10, 11	36
2.5 Proof of Theorem 2.1.1 in genus 12, 13	38
Bibliography	50
3 Stability of rank-3 Lazarsfeld-Mukai bundles on $K3$ surfaces	51
3.1 Introduction and statement of the results	51
3.2 Linear systems on $K3$ surfaces	55
3.3 Lazarsfeld-Mukai bundles	56
3.4 Mumford stability for sheaves on $K3$ surfaces	57
3.5 Stability of Lazarsfeld-Mukai bundles of rank 2	59
3.6 Lazarsfeld-Mukai bundles of rank 3 which are not μ_L -stable	63
3.7 Cases with a μ_L -stable subbundle of rank 2 and L -slope $\geq \mu_L(E)$	68
3.8 Cases with a μ_L -stable quotient sheaf of rank 2 and L -slope $\leq \mu_L(E)$	70
3.9 Remaining cases	73
3.10 Transversality of some Brill-Noether loci	77

Contents

3.11	Noether-Lefschetz divisor and Gieseker-Petri divisor in genus 11	82
	Bibliography	88
4	Green's Conjecture for curves on Del Pezzo surfaces	89
4.1	Introduction	89
4.2	Syzygies and Koszul Cohomology	91
4.3	Linear systems on Del Pezzo surfaces	92
4.4	The analogue of the Lazarsfeld-Mukai bundle	94
4.5	Parameter count	96
4.6	Proof of Theorem 4.1.1	99
4.7	Further remarks	100
	Bibliography	102

1 Introduction

The study of moduli spaces is one of the leading research directions in contemporary algebraic geometry. The word “moduli” was first used by Riemann in 1857 (cf. [56]) in order to refer to the parameters for a certain class of varieties. He proved that compact complex Riemann surfaces of genus $g \geq 2$ depend on $3g - 3$ parameters. However, the first rigorous construction of the moduli space M_g as an analytic variety, which parametrizes isomorphism classes of smooth, irreducible curves of genus g , was accomplished only in 1940 by Teichmüller (cf. [63]). In 1965, by means of Geometric Invariant Theory, Mumford performed a completely different construction of M_g as a quasiprojective algebraic variety valid in any characteristic (cf. [50]). Since any smooth curve of genus g can be embedded in \mathbb{P}^{5g-6} via the tricanonical map, the moduli space M_g can be realized as the GIT quotient of a locally closed subset of the Hilbert scheme of smooth curves of degree $6g - 6$ and genus g in \mathbb{P}^{5g-6} by the algebraic group $PGL(5g - 5, \mathbb{C})$. As remarked by Mumford, the presence of curves with non-trivial automorphisms prevents the existence of a universal family on the scheme M_g , which is therefore just the *coarse moduli space* of a more fundamental object, the *moduli stack* \mathcal{M}_g . Since the group of automorphisms of any smooth, irreducible curve of genus $g \geq 2$ is finite, the space \mathcal{M}_g is a so-called Deligne-Mumford stack. In [18] Deligne and Mumford introduced the compactification \overline{M}_g of M_g by allowing curves to degenerate to stable ones. A curve is stable if it is complete and connected, its only singularities are nodes and its automorphism group is finite.

Surprisingly enough, some properties of M_g were determined even before it was constructed. For instance, already in 1882 Klein (cf. [41]), following some ideas of Clebsch, proved the irreducibility of M_g . This is a consequence of the Riemann’s Existence Theorem since the space parametrizing branched coverings of \mathbb{P}^1 of fixed degree and genus is connected.

Since the end of the nineteenth century, the development of the study of special divisors on algebraic curves, under the name of Brill-Noether theory, somehow presumed the existence of M_g . Indeed, in many statements, such as the Brill-Noether Theorem, one encounters the notion of “being general in moduli”. It is not surprising that the proofs of some very early results were originally incomplete and were accomplished only after the work of Deligne and Mumford.

Since \overline{M}_g is irreducible and projective, a big advantage of considering curves in families is that some conditions can be verified on a single curve in order to obtain their validity for “almost every” curve, namely in a dense open subset of the moduli space. In fact, degeneration to specific types of singular curves proved very successful for some fundamental theorems in Brill-Noether theory. However, the study of line bundles on singular curves is not easy and generally requires sophisticated combinatorial

1 Introduction

techniques. The major problem faced while adopting this strategy is that the Picard functor of a singular curve is in general neither complete nor separated.

More recently, Lazarsfeld considered specializations to smooth curves lying on certain types of surfaces. In particular, he proved that sections of general $K3$ surfaces behave generically from the point of view of special divisors. The Brill-Noether theory of curves on a $K3$ surface S can be investigated by looking at certain vector bundles on S . The study of the geometry of moduli spaces of sheaves on this type of surfaces is a very active research field in itself.

This introduction is aimed to motivate the problems studied in the following chapters, which mainly concern Brill-Noether theory of algebraic curves in general and of $K3$ -sections in particular, and to place them into context. By giving an overview of the subject, without any claim of being exhaustive, I intend to explain why Gieseker-Petri loci are interesting and how they are related to curves lying on $K3$ surfaces. I will omit most of the proofs, especially those involving degeneration to singular curves since this strategy is never performed in the rest of the thesis. On the contrary, I will fill details while recalling some results on $K3$ -sections, which play a central role in Chapter 3.

1.1 The Brill-Noether Theorem

Let C be a smooth, irreducible curve of genus g . A linear series of type g_d^r on C is a pair (A, V) such that $A \in \text{Pic}^d(C)$ and $V \subset H^0(C, A)$ is an $(r + 1)$ -dimensional subspace. When $V = H^0(C, A)$, the linear series is said to be *complete*. An effective divisor D on C is *special* if

$$h^0(C, \mathcal{O}_C(D)) > d - g + 1;$$

Riemann-Roch Theorem and Serre duality imply that this is equivalent to require that $h^0(C, \mathcal{O}_C(K_C - D)) > 0$ and force the degree to be in the range $0 \leq d \leq 2g - 2$.

Examples of special divisors had already appeared in Riemann's paper concerning abelian functions (cf. [56]). However, a methodical study of special linear series on algebraic curves was started in 1874 by the German geometers Brill and Noether (cf. [9]), after whom the theory was named. The problem of studying curves embedded in a projective space can be translated into that of understanding their linear series. Indeed, any very ample linear series of type g_d^r on a curve C naturally defines an embedding of C as a degree- d curve in \mathbb{P}^r . In [9] the following result was first stated, which became known as the Brill-Noether Theorem.

Theorem 1.1.1. *A general curve $[C] \in M_g$ has a g_d^r if and only if the Brill-Noether number*

$$\rho(g, r, d) := g - (r + 1)(g - d + r)$$

is nonnegative.

I sketch Brill and Noether's plausibility argument. Consider the canonical embedding $C \hookrightarrow \mathbb{P}^{g-1}$; by Riemann-Roch Theorem, C has a g_d^r whenever there exists an r -

dimensional family of $(d - r - 1)$ -planes in \mathbb{P}^{g-1} which are d -secant to C . The number of conditions imposed on a plane $\pi \in G(d - r - 1, g - 1)$ by forcing it to intersect C in a point equals $(g - d + r - 1)$. Hence, the locus $G_1 \subset G(d - r - 1, g - 1)$, consisting of the planes which are d -secant to C , has codimension at most $d(g - d + r - 1)$, if nonempty. The condition

$$\dim G(d - r - 1, g - 1) - d(g - d + r - 1) \geq r$$

is equivalent to the inequality $\rho(g, r, d) \geq 0$. However, this argument only proves that, if $\rho(g, r, d) \geq 0$ and $W_d^r(C) \neq \emptyset$, then every irreducible component of $W_d^r(C)$ has dimension $\geq \rho(g, r, d)$.

I will give a brief excursus of the developments of the Brill-Noether theory until the problems in providing a rigorous proof of the Brill-Noether Theorem were finally worked out.

One of the earliest achievements is due to Clifford ([16]), who proved that any complete linear series A of type g_d^r on a smooth curve C satisfies $d \geq 2r$; if equality holds, then C is hyperelliptic or D is equivalent to either 0 or K_C .

Castelnuovo, Enriques and Severi obtained meaningful results concerning embeddings of curves in projective spaces, such as the General Position Theorem ([59]) and Castelnuovo's Bound ([10]). I also want to mention the Enriques-Babbage Theorem ([26, 7]) stating that a canonical curve $C \subset \mathbb{P}^{g-1}$, which is not isomorphic to a plane quintic and has no linear series of type g_3^1 , is cut out by quadrics.

However, the proof of the existence part of the Brill-Noether Theorem, actually valid for any curve $[C] \in M_g$, was obtained quite late and is due to Kleiman and Laksov ([40]) and Kempf ([39]), who introduced the Brill-Noether variety $W_d^r(C)$, parametrizing complete linear series of degree d and dimension at least r on C .

Indeed, let D be a fixed divisor of sufficiently high degree m on C and \mathcal{L} be a Poincaré line bundle on $C \times \text{Pic}^{m+d}(C)$, whose restriction to the fibre of the projection morphism $\nu : C \times \text{Pic}^{m+d}(C) \rightarrow \text{Pic}^{m+d}(C)$ over a point $A \in \text{Pic}^{m+d}(C)$ coincides with A itself. Set $\Gamma := D \times \text{Pic}^{m+d}(C)$ and consider the isomorphism $a : \text{Pic}^d(C) \rightarrow \text{Pic}^{m+d}(C)$ given by mapping a point L to $L(D)$. Then, $W_d^r(C)$ is the pullback to $\text{Pic}^d(C)$ under a of the $(m + d - g - r)$ -th determinantal variety attached with the evaluation map

$$e : \nu_* \mathcal{L} \longrightarrow \nu_* (\mathcal{L} / \mathcal{L}(-\Gamma)).$$

For $m + d \geq 2g - 1$, the Riemann-Roch Theorem implies that e is a morphism between two vector bundles of rank $m + d - g + 1$ and m respectively. In particular, the expected dimension of $W_d^r(C)$ equals the Brill-Noether number $\rho(g, r, d)$. When $\rho(g, r, d) \geq 0$, the non-emptiness of $W_d^r(C)$ follows from a more general result of Bloch and Gieseker (cf. [8]), by showing that the dual bundle of $\nu_* \mathcal{L}$ is ample if $m + d \geq 2g - 1$. Notice that a similar argument implies the connectedness of $W_d^r(C)$ when $\rho(g, r, d) \geq 1$, as proved by Fulton and Lazarsfeld in [29]. I also mention that the variety $G_d^r(C)$, which parametrizes linear series of type g_d^r on C , was later defined by Arbarello and Cornalba (cf. [4]) as the canonical blow-up of $W_d^r(C)$ along $W_d^{r+1}(C)$.

The non-existence part of the Brill-Noether Theorem was proved by Griffiths and

1 Introduction

Harris in [34] by means of some degeneration techniques. The strategy had already been indicated by Castelnuovo and Severi. Since the statement concerns curves in an open and dense subset of M_g , it is enough to prove it for just one curve of genus g . Because of the difficulties in explicitly exhibiting smooth curves which are not very special with respect to the existence of linear series, one has to “move” curves in families. Let $\mathcal{C} \rightarrow S$ be a family of genus g curves such that the fibre C_t is smooth and irreducible if $t \neq 0$ while the central fibre C_0 is singular; one can check the statement on C_0 and then deduce its validity for a general fibre C_t . This second step is non-trivial since in general the family $\{\text{Pic}^d(C_t)\}$ is not proper. Kleiman understood that such a problem can be solved by considering, instead of $\text{Pic}^d(C_t)$, the variety parametrizing torsion free sheaves of rank 1 on C_t ; these correspond to line bundles on some partial normalization of the curve.

In Severi and Kleiman’s attempts, the special fibre C_0 is a g -nodal curve, that is, a rational curve obtained by identifying g pairs of points $\{P_i, Q_i\}$ on \mathbb{P}^1 . Griffiths and Harris’ successful idea was to introduce another degeneration, that is, let the points Q_1, P_2, Q_2, \dots tend to P_1 one after the other. Later on, Eisenbud and Harris provided an easier proof by reduction to the case of g -cuspidal curves (cf. [21]).

A remarkable consequence of the Brill-Noether Theorem is that one may investigate the existence of some special linear series on a curve C , in order to understand how special C is in moduli. In fact, when $\rho(g, r, d) < 0$, one defines the Brill-Noether locus

$$M_{g,d}^r := \{[C] \in M_g \mid W_d^r(C) \neq 0\},$$

which is a subvariety of codimension at least 1 in M_g . A result of Eisenbud and Harris (cf. [24]), together with a theorem of Steffen (cf. [61]), implies that the locus $M_{g,d}^r$ is an irreducible divisor in M_g if $\rho(g, r, d) = -1$, while it has codimension at least 2 if $\rho(g, r, d) \leq -2$. Further results concerning the dimension of the loci $M_{g,d}^r$ will be recalled in Chapter 2.

1.2 The Gieseker-Petri Theorem

Let A be a line bundle on a curve $[C] \in M_g$. The Petri map $\mu_{0,A}$ is, by definition, the cup-product homomorphism

$$\mu_{0,A} : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee) \rightarrow H^0(C, \omega_C).$$

The Petri map $\mu_{0,V}$ associated with a non-complete linear series (A, V) is the restriction of $\mu_{0,A}$ to the subspace $V \otimes H^0(C, \omega_C \otimes A^\vee)$. If (A, V) is of type g_d^r , the Riemann-Roch Theorem implies that the Brill-Noether number $\rho(g, r, d)$ is the difference between the dimensions of the domain of $\mu_{0,V}$ and of its codomain.

The determinantal description of $W_d^r(C)$, together with the isomorphism between $H^1(C, \mathcal{O}_C)$ and the tangent space to $\text{Pic}(C)$ in any point, implies that the tangent space

to $W_d^r(C)$ in a point $A \notin W_d^{r+1}(C)$ is the kernel of the map

$$\psi_A : H^1(C, \mathcal{O}_C) \rightarrow \text{Hom}(H^0(C, A), H^1(C, A)).$$

Since the dual of ψ_A coincides the Petri map $\mu_{0,A}$, it follows that

$$T_A(W_d^r(C)) = (\text{Im } \mu_{0,A})^\perp.$$

Hence, every irreducible component of $W_d^r(C)$ has dimension $\geq \rho(g, r, d)$ and $W_d^r(C)$ is smooth in A of dimension $\rho(g, r, d)$ whenever $\mu_{0,A}$ is injective. Analogously, the requirement for $G_d^r(C)$ to be smooth of the expected dimension in a point (A, V) is equivalent to the condition $\ker \mu_{0,V} = 0$.

The following result was indirectly stated by Petri ([54]) and first proved by Gieseker ([31]) by considering the stable reduction of the family of curves used by Griffiths and Harris in order to prove the Brill-Noether Theorem.

Theorem 1.2.1. *If $[C] \in M_g$ is general, the Petri map $\mu_{0,A}$ is injective for any line bundle $A \in \text{Pic}(C)$.*

In particular, if $\rho(g, r, d) \geq 0$, then $G_d^r(C)$ is smooth of the expected dimension $\rho(g, r, d)$ and the forgetful map $G_d^r(C) \rightarrow W_d^r(C)$ is a rational resolution of singularities along $\text{Sing } W_d^r(C) = W_d^{r+1}(C)$. Also notice that the Gieseker-Petri Theorem trivially implies the non-existence part of the Brill-Noether Theorem.

A simpler proof of the Gieseker-Petri Theorem is due to Eisenbud and Harris (cf. [20]). Their idea was to specialize to g -cuspidal rational curves, for which they had already proved the validity of the Brill-Noether Theorem. Since Petri's condition can fail for such curves, another degeneration was necessary, namely, they let the cusps come together one after the other. They considered the stable reduction of the family; possibly after further blow-ups and base changes, they obtained a flag curve C_0 , consisting of a chain of rational curves X_1, \dots, X_n such that to g of the X_i an elliptic tail is joined possibly by another rational chain.

The study of line bundles on such a curve C_0 is possible thanks to the theory of *limit linear series* (cf. [22]). This enables to control the degeneration of linear series on smooth curves while such curves degenerate to a nodal curve C_0 of compact type, that is, the dual graph of C_0 is a tree or, equivalently, the Jacobian $\text{Pic}^0(C_0)$ is compact.

In section 1.4, I will recall Lazarsfeld's proof of the Gieseker-Petri Theorem by means of curves lying on a $K3$ surface.

Having defined the Gieseker-Petri locus in genus g as

$$GP_g := \{[C] \in M_g \mid C \text{ does not satisfy the Gieseker-Petri Theorem}\},$$

the Gieseker-Petri Theorem can be rephrased in the following way.

Theorem 1.2.2. *The Gieseker-Petri locus GP_g has codimension at least 1 in M_g .*

Notice that GP_g naturally breaks up in different components depending on the numerical type of the linear series for which the Gieseker-Petri Theorem fails. For fixed

1 Introduction

values of r and d , one defines the Gieseker-Petri locus of type (r, d) as

$$GP_{g,d}^r := \{[C] \in M_g \mid \exists (A, V) \in G_d^r(C) \text{ with } \ker \mu_{0,V} \neq 0\};$$

if $\rho(g, r, d) < 0$, then $GP_{g,d}^r$ trivially coincides with the Brill-Noether locus $M_{g,d}^r$.

1.3 M_g is of general type for $g \geq 24$

Brill-Noether loci and Gieseker-Petri loci played a central role in the proof of the following result, due to Harris and Mumford.

Theorem 1.3.1. *The coarse moduli space M_g is of general type for $g \geq 24$.*

Recall that a smooth projective variety X is of *general type* whenever the canonical divisor K_X is big, that is, the n -canonical map

$$\phi_{nK_X} : X \dashrightarrow \mathbb{P}H^0(X, nK_X)$$

is generically injective for n sufficiently high. Harris and Mumford's Theorem was very unexpected since it invalidated Severi's Conjecture, which predicted the unirationality of M_g in any genus and had been proved for genus $g \leq 10$ by Severi himself. The proof is based on a divisor class computation.

I remind that, for $g \geq 3$, the group $\text{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ is freely generated by the classes $\lambda, \delta_0, \dots, \delta_{\lfloor g/2 \rfloor}$, where λ is the first Chern class of the Hodge bundle \mathbb{E} , whose fibre over a point $[C] \in M_g$ is the space $H^0(C, \omega_C)$, and δ_i are the classes of the boundary divisors Δ_i . Here, Δ_0 denotes the closure of the locus of irreducible curves having one node while, if $i \geq 1$, a general point of Δ_i has two components, of genus i and $g - i$, meeting in a point. If $D \subset \overline{M}_g$ coincides with the closure of an effective divisor on M_g , then $[D] = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$ for some positive integers a and b_i and one defines the *slope* of D as

$$s(D) := \frac{a}{\min_{i=0}^{\lfloor g/2 \rfloor} b_i}.$$

The computation of the canonical class

$$K_{\overline{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \dots - 2\delta_{\lfloor g/2 \rfloor}$$

was performed by Harris and Mumford and exploits Kodaira-Spencer theory and the Grothendieck-Riemann-Roch Theorem. Since the Hodge class λ is big (cf. [49]), the same holds true for $K_{\overline{M}_g}$ provided there exists a divisor D with slope $< 13/2$; indeed, this ensures that $[K_{\overline{M}_g}]$ lies in the cone spanned by λ , $[D]$ and δ_i for $0 \leq i \leq \lfloor g/2 \rfloor$.

In the cases of odd genus $g = 2k + 1$ (cf. [36]), Harris and Mumford considered the closure in \overline{M}_g of the locus $D_k \subset M_g$ consisting of curves C admitting a mapping $\varphi : C \xrightarrow{k:1} \mathbb{P}^1$. Since $\rho(g, 1, k) = -1$, the locus D_k is an open and dense subscheme of the irreducible divisor $M_{g,k}^1$, the difference $M_{g,k}^1 \setminus D_k$ consisting of those curves for which

1.4 Lazarsfeld's proof of the Gieseker-Petri Theorem

every g_k^1 has some base points; as a consequence, the closure $\overline{D_k}$ inside $\overline{M_g}$ coincides with $\overline{M_{g,k}^1}$. It turned out that

$$s(\overline{M_{g,k}^1}) = 6 + \frac{12}{g+1},$$

which is less than $13/2$ for $g \geq 24$.

The cases of even genus $g = 2k - 2$ were first treated by Harris (cf. [35]) when $g \geq 40$ and later on by Eisenbud and Harris ([23]), who simplified the proof and improved the result up to genus $g \geq 24$ by means of limit linear series. The role of the divisor D_k^1 is played by the closure of the branch divisor of the forgetful map $H_{g,k} \rightarrow M_g$ from the Hurwitz scheme $H_{g,k}$ parametrizing coverings of \mathbb{P}^1 of degree k having as source a smooth curve C of genus g . This coincides with the locus

$${}^bGP_{g,k}^1 := \left\{ [C] \in M_g \mid \exists \text{ a base point free } (A, V) \in G_k^1(C) \text{ with } \ker \mu_{0,V} \neq 0 \right\},$$

which is open but not trivially dense in $GP_{g,k}^1$.

In [23] it was also proved that the classes of all the Brill-Noether divisors contained in M_g only differ by a positive rational factor. On the contrary, the classes of the codimension-1 components of the loci $GP_{g,d}^r$ for $\rho(g, r, d) \geq 0$ have been computed only in few cases, for instance if $\rho(g, r, d) \in \{0, 1\}$ (cf. [27, 28]).

1.4 Lazarsfeld's proof of the Gieseker-Petri Theorem

In [45] Lazarsfeld gave an easier proof of the Gieseker-Petri Theorem. As already remarked, the only smooth curves which can be explicitly written down are very special from the point of view of Brill-Noether theory. Gieseker, Eisenbud and Harris had overcome this problem by specializing to certain types of singular curves. Lazarsfeld understood how to prove the statement directly for some smooth curves, thus avoiding degenerations. His idea was to consider an integral curve C_0 of genus g lying on a K3 surface S and let it move in its linear system $|L|$, where $L = \mathcal{O}_S(C_0)$. Notice that, when g is sufficiently high, the curve C_0 is far from being general in moduli; indeed, curves lying on K3 surfaces depend on $19 + g$ parameters if $g = 11$ or $g \geq 13$ (cf. [14]). Lazarsfeld proved the following.

Theorem 1.4.1. *Let S be a K3 surface such that $\text{Pic}(S) = \mathbb{Z} \cdot L$. Then, a general curve $C \in |L|$ satisfies the Gieseker-Petri Theorem.*

The same holds true under the weaker hypothesis that every curve in the linear system $|L|$ is reduced and irreducible. Since for any $g \geq 2$ there exists a K3 surface whose Picard group is generated by the class of a smooth, irreducible curve of genus g , Lazarsfeld's result trivially implies the Gieseker-Petri Theorem. I am going to recall the proof of Theorem 1.4.1, which was somehow simplified by Pareschi in [53].

Let C be a smooth and irreducible curve in $|L|$. Given a complete and base point free $A \in W_d^r(C)$, one can think of A as a globally generated sheaf on S and define an

1 Introduction

$(r+1)$ -rank vector bundle $F_{C,A}$ on S to be the kernel of the evaluation map

$$\mathrm{ev}_{A,S} : H^0(C, A) \otimes \mathcal{O}_S \rightarrow A.$$

Let $M_{C,A}$ denote the bundle on C of rank r obtained as kernel of the map $\mathrm{ev}_{A,C}$ given by evaluating the sections of A on \mathcal{O}_C , that is, $M_{C,A}$ fits in the following short exact sequence:

$$0 \rightarrow M_{C,A} \rightarrow H^0(C, A) \otimes \mathcal{O}_C \rightarrow A \rightarrow 0.$$

After tensoring with $\omega_C \otimes A^\vee$ and taking global sections, one obtains that

$$H^0(S, M_{C,A} \otimes \omega_C \otimes A^\vee) = \ker \mu_{0,A}. \quad (1.1)$$

Notice that $\det F_{C,A} = L^\vee$, while $\det M_{C,A} = A^\vee$. Since $\omega_C = \mathcal{O}_C(C)$ by the adjunction formula, one has a short exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow F_{C,A} \otimes \omega_C \otimes A^\vee \rightarrow M_{C,A} \otimes \omega_C \otimes A^\vee \rightarrow 0.$$

Pareschi proved that, for a general choice of (C, A) , this sequence is exact on the global sections. Indeed, denote by $|L|_s$ the locus of smooth curves in $|L|$ and consider the projection $\pi : \mathcal{W}_d^r(|L|) \rightarrow |L|_s$, whose fibre over C coincides with the Brill-Noether variety $W_d^r(C)$. Let

$$\mu_{1,A} : \ker \mu_{0,A} \rightarrow H^0(C, \omega_C^{\otimes 2})$$

be the Gaussian map and

$$\rho^\vee : H^0(C, \omega_C^{\otimes 2}) \rightarrow (T_C|L|)^\vee = H^1(C, \mathcal{O}_C)$$

be the transpose of the Kodaira-Spencer map. It turns out that the coboundary map $\delta : H^0(C, M_{C,A} \otimes \omega_C \otimes A^\vee) \rightarrow H^1(C, \mathcal{O}_C)$ coincides, up to a scalar factor, with the composition $\mu_{1,A,S} := \rho^\vee \circ \mu_{1,A}$. Moreover, one shows that

$$\mathrm{Im}(d\pi_{(C,A)}) \subset \mathrm{Ann}(\mathrm{Im}(\mu_{1,A,S})) = \mathrm{Ann}(\mathrm{Im} \delta).$$

Therefore, if C is general in its linear system, Sard's Lemma implies that $\delta \equiv 0$, that is, one finds

$$0 \rightarrow H^0(C, \mathcal{O}_C) \rightarrow H^0(C, F_{C,A} \otimes \omega_C \otimes A^\vee) \rightarrow H^0(C, M_{C,A} \otimes \omega_C \otimes A^\vee) \rightarrow 0. \quad (1.2)$$

Set $E_{C,A} = F_{C,A}^\vee$. By the very definition, $E_{C,A}$ fits into the short exact sequence

$$0 \rightarrow H^0(C, A)^\vee \otimes \mathcal{O}_S \rightarrow E_{C,A} \rightarrow \omega_C \otimes A^\vee \rightarrow 0; \quad (1.3)$$

hence, $h^i(S, E_{C,A}) = 0$ for $i \in \{1, 2\}$ and $E_{C,A}$ is globally generated off the base points of $\omega_C \otimes A^\vee$. Tensoring (1.3) by $F_{C,A}$ and taking cohomology, one finds that

$$H^0(S, E_{C,A} \otimes F_{C,A}) \simeq H^0(C, F_{C,A} \otimes \omega_C \otimes A^\vee). \quad (1.4)$$

1.5 Propagation of linear series on K3-sections

Putting (1.2), (1.1), (1.4) together, one obtains the following result.

Proposition 1.4.2. *If $C \in |L|$ is general, then for any base point free $A \in \text{Pic}(C)$ one has:*

$$\ker \mu_{0,A} = 0 \iff E_{C,A} \text{ is simple.}$$

Notice that Riemann-Roch Theorem implies

$$\chi(S, E_{C,A} \otimes E_{C,A}^\vee) = 2(1 - \rho(g, r, d));$$

hence, if $\rho(g, r, d) < 0$, the Lazarsfeld-Mukai bundle corresponding to any linear series A of type g_d^r on a curve $C \in |L|_s$ is non-simple, even without assuming that C is general in its linear system.

It remains to prove that the hypotheses of Theorem 1.4.1 force all the bundles of type $E_{C,A}$ to be simple, that is, to have no non-trivial endomorphisms. Ab absurdo assume that $h^0(S, E_{C,A} \otimes E_{C,A}^\vee) \geq 2$ and let φ be an endomorphism of $E_{C,A}$ which is not a multiple of the identity. Set $\psi := \varphi - \lambda \cdot \text{Id}_{E_{C,A}}$, where λ is an eigenvalue of $\varphi(s)$ for some $s \in S$. Since $\det \psi$ vanishes in s and $H^0(S, L \otimes L^{-1}) = H^0(S, \mathcal{O}_S) = \mathbb{C}$, we have $\det \psi \equiv 0$ and, up to replacing φ with ψ , we can assume that φ drops rank everywhere on S . Therefore, both $N := \text{Im} \varphi$ and $M := \text{Coker} \varphi$ have positive rank. Having denoted by $T(M)$ the torsion subsheaf of M and set $M_0 := M/T(M)$, we obtain

$$[C] = c_1(L) = c_1(N) + c_1(M_0) + c_1(T(M)).$$

Since both N and M_0 are quotient of $E_{C,A}$, they are globally generated off a finite set. This ensures that $c_1(N)$ and $c_1(M_0)$ are represented by some effective divisors on S and are non-trivial because $H^2(S, E_{C,A}) = 0$. Since $c_1(T(M))$ is a non-negative linear combination of the components of $\text{supp } T(M)$ of codimension 1 in C , we have shown that $c_1(L)$ is the sum of two effective divisors. This contradicts the hypothesis that $\text{Pic}(S)$ is generated by L and we are done.

The bundle $E_{C,A}$ is commonly known as the *Lazarsfeld-Mukai bundle* associated with the pair (C, A) . Thanks to Proposition (1.4.2), many Brill-Noether theoretic problems for curves on a K3 surface S can be translated in terms of the corresponding Lazarsfeld-Mukai bundles. In fact, the projections $\pi : \mathcal{W}_d^r(|L|) \rightarrow |L|_s$ can be studied by looking at the geometry of certain moduli spaces of sheaves on S .

1.5 Propagation of linear series on K3-sections

I recall the definition of some invariants of a smooth curve C .

The *gonality* of C is the minimum degree d such that $G_d^1(C) \neq \emptyset$; the Brill-Noether Theorem implies that, if C has genus g , then $\text{gon}(C) \leq \lfloor (g+3)/2 \rfloor$. If a line bundle A on C satisfies both $h^0(C, A) \geq 2$ and $h^1(C, A) \geq 2$, set

$$\text{Cliff}(A) := \deg(A) - 2h^0(C, A) + 2 \geq 0.$$

1 Introduction

The *Clifford index* of C is defined as

$$\text{Cliff}(C) := \min\{\text{Cliff}(A) \mid A \in \text{Pic}(C), h^i(C, A) \geq 2 \text{ for } i = 0, 1\}. \quad (1.5)$$

We say that $A \in \text{Pic}(C)$ *contributes* to $\text{Cliff}(C)$ if it satisfies the inequalities in (1.5), and that it *computes* $\text{Cliff}(C)$ if $\text{Cliff}(A) = \text{Cliff}(C)$. The *Clifford dimension* of C is, by definition, the minimum integer $r \geq 1$ such that there exists a line bundle A on C which computes the Clifford index and such that $h^0(C, A) = r + 1$. Curves of Clifford dimension ≥ 2 are rare. For instance, equality holds only for smooth plane curves. Coppens and Martens ([17]) proved that, if C has Clifford dimension ≥ 2 and gonality k , then $\text{Cliff}(C) = k - 3$ and C has a 1-dimensional family of g_k^1 .

Let us now focus on the Brill-Noether theory of curves on $K3$ surfaces. A *K3-section* is, by definition, a smooth irreducible curve of genus at least 2 lying on a $K3$ surface. In [57], Saint-Donat proved that, if a $K3$ -section $C \subset S$ has either a g_2^1 or a g_3^1 , the same happens for any curve $C' \in |L|$, where $L := \mathcal{O}_S(C)$. Reid ([55]) extended this result by showing that, if $g > d^2/4 + d + 2$, any g_d^1 on C propagates since it is cut out by an elliptic pencil $|E|$ on S . This suggests that the existence of some special linear series on a $K3$ section $C \subset S$ might depend only on the linear equivalence class of C in S . Such an idea originated also from the work of Harris and Mumford [36], who conjectured that the gonality remains constant while moving C in its linear system $|L|$.

Donagi and Morrison disproved Harris and Mumford's Conjecture by exhibiting the following counterexample (cf. [19]). Let $\pi : S \rightarrow \mathbb{P}^2$ be a $K3$ surface which is a double cover of \mathbb{P}^2 branched along a smooth plane sextic and let $L := \pi^* \mathcal{O}_{\mathbb{P}^2}(3)$. One can check that $L^2 = 18$; this ensures that the projective dimension of the linear system $|L|$ is 10 and equals the genus of any smooth curve $C \in |L|_S$. Curves in the subspace $|\pi^* H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))|$, which has codimension 1 in $|L|$, are double covers of an elliptic curve, thus they have gonality 4. On the other hand, a general element in $|L|$ is isomorphic to a smooth plane sextic, hence its gonality is 5. However, both the pencils g_4^1 on a curve inside $|\pi^* H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))|$ and the g_5^1 on a general curve in $|L|$ are contained in a linear series of type g_6^2 , which computes the Clifford index and is induced from the same line bundle $\pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ on S .

Donagi and Morrison's example suggests that the Clifford index, instead of the gonality, might be constant for all curves in $|L|$. This was conjectured by Green (cf. [33]) in connection with his theory of Koszul cohomology. Indeed, he noticed that in the ample case such a statement would be implied by a conjecturally generalization of the Enriques-Babbage Theorem, now known as the Green's Conjecture. Moreover, one can expect that any g_d^r on C such that $\rho(g, r, d) < 0$ is contained in the restriction to C of a line bundle on S .

In [19] Donagi and Morrison showed that, if C has Clifford dimension 1, then $\text{Cliff}(C')$ equals $\text{Cliff}(C)$ for any other smooth curve $C' \in |L|$. They actually proved a somehow stronger statement, which we are now going to recall. We say that $M \in \text{Pic}(S)$ is *adapted to* $|L|$ if the following conditions are satisfied:

$$(i) \quad h^0(S, M) \geq 2, h^0(S, L \otimes M^\vee) \geq 2,$$

(ii) $h^0(C, M \otimes \mathcal{O}_C)$ is independent of the curve $C \in |L|_S$.

This ensures that the restriction of M to a smooth curve $C \in |L|$ contributes to the Clifford index and $\text{Cliff}(M \otimes \mathcal{O}_C)$ does not depend on the choice of $C \in |L|_S$. Donagi and Morrison proved the following.

Theorem 1.5.1. *Let S be a K3 surface and $L \in \text{Pic}(S)$ be a line bundle such that curves in $|L|$ are non-hyperelliptic of genus g . Let A be a complete, base point free g_d^1 on a smooth curve $C \in |L|$ such that $\rho(g, 1, d) < 0$.*

Then, there exists $M \in \text{Pic}(S)$ adapted to $|L|$ such that the linear system $|A|$ is contained in $|M \otimes \mathcal{O}_C|$ and $\text{Cliff}(M \otimes \mathcal{O}_C) \leq \text{Cliff}(A)$.

I recall the idea of the proof. Let $E := E_{C,A}$ be the Lazarsfeld-Mukai bundle associated with the pair (C, A) . The hypothesis $\rho(g, 1, d) < 0$ prevents E from being simple, thus there exists a non-zero endomorphism $\varphi : E \rightarrow E$ which drops rank everywhere on S . Let $N = \ker \varphi$ and $M \otimes I_\xi = \text{Im} \varphi$, where $N, M \in \text{Pic}(S)$ and I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$. The bundle E sits in the following short exact sequence:

$$0 \rightarrow N \rightarrow E \rightarrow M \otimes I_\xi \rightarrow 0, \quad (1.6)$$

which is known as the *Donagi-Morrison extension*. Since E is globally generated off a finite set and $H^2(S, E) = 0$, then M is base point free and non-trivial, thus $h^0(S, M) \geq 2$. Moreover, either $E \simeq N \oplus M$ or φ is nilpotent by a theorem of Atiyah (cf.[6]). We assume that (1.6) does not split, the case where E is the direct sum of two line bundles being easier. Since $\text{Im} \varphi \subset \ker \varphi$, one has $h^0(S, M^\vee \otimes N) \neq 0$. It follows that $h^0(S, N)$ is greater or equal than 2 and (i) is satisfied.

The bundle E naturally has a section s such that, if $(s) = A_0$, then $A = \mathcal{O}_C(A_0)$. Since s is not a section of N , one finds that $A_0 \subset D$ where $D = c_1(M)$. It is not difficult to check that $\text{Cliff}(M \otimes \mathcal{O}_C) \leq d - 2$. Finally, condition (ii) follows from the fact that $h^1(S, M) = 0$ unless $D \equiv k\Gamma$ for some positive integer k and some elliptic curve $\Gamma \subset S$; in this case, Donagi and Morrison explain how to replace D with a divisor \tilde{D} which is adapted to $|L|$ and has the same properties as D .

Donagi and Morrison conjectured that an analogue of Theorem 1.5.1 should hold for any g_d^r on a curve $C \subset S$ such that $\rho(g, r, d) < 0$. In particular, such a g_d^r is expected to be contained in a linear series g_e^s on C , which is induced from a line bundle on S , contributes to the Clifford index of C and satisfies $e - 2s \leq d - 2r$. However, their proof cannot be adapted to the cases where $r \geq 2$ and E has rank at least 3, hence their conjecture remains open.

In [32] Green and Lazarsfeld proved that linearly equivalent curves on a K3 surface have the same Clifford index, independently on their Clifford dimension. More precisely, their result states the following.

Theorem 1.5.2. *Let S be a K3 surface and $C \subset S$ be a smooth irreducible curve of genus $g \geq 2$. Then*

$$\text{Cliff}(C') = \text{Cliff}(C)$$

1 Introduction

for every curve $C' \in |L|_s$, where $L := \mathcal{O}_S(C)$. Moreover, if $\text{Cliff}(C) < \lfloor (g-1)/2 \rfloor$, there exists a line bundle $M \in \text{Pic}(S)$ whose restriction to any curve $C' \in |L|_s$ computes the Clifford index of C' .

The argument is as follows. Assume that C has the minimal Clifford index among all the smooth curves in the same linear system $|L|$, and let A be a line bundle on C computing $\text{Cliff}(C)$ such that $\deg A \leq g-1$. One would like to obtain the line bundle M as a subsheaf of $E_{C,A}$; this turns out to be possible up to replacing $E_{C,A}$ with what Green and Lazarsfeld call a *reduction of minimal rank*. If S does not contain any smooth rational curve, such a replacement is equivalent to the further assumption that C has minimal Clifford dimension r among all the curves in $|L|$ having the same Clifford index as C , and $h^0(C, A) = r+1$.

If $r = 1$, one proceeds as Donagi and Morrison. If $r \geq 2$, one uses the inequality $h^0(S, E_{C,A}) \geq 2\text{rk } E_{C,A}$ in order to prove the existence of $s \in H^0(S, E_{C,A})$ vanishing in at least two points. The section s vanishes along a divisor D because otherwise there would exist a pair (C', A') , satisfying $C' \in |L|_s$, $A' \in \text{Pic}(C')$, $\text{Cliff}(A') \leq \text{Cliff}(A)$ and $h^0(C', A') < h^0(C, A)$. If S does not contain any smooth elliptic curve, one shows that the restriction of $M := \mathcal{O}_S(D)$ to any curve $C' \in |L|_s$ contributes to the Clifford index of C' and $\text{Cliff}(M \otimes \mathcal{O}_{C'}) \leq \text{Cliff}(A)$. Such a construction of M deeply uses the minimality assumptions on $\text{Cliff}(A)$ and cannot be performed in order to prove Donagi and Morrison's Conjecture for arbitrary g_d^r with $\rho(g, r, d) < 0$.

1.6 Gonality and Clifford dimension of K3-sections

Quite surprisingly, Donagi and Morrison's example turned out to be the only case where Harris and Mumford's Conjecture fails. Observe that in this example the curves of minimal gonality 4 have a 1-dimensional family of g_4^1 since they are double covers of an elliptic curve. This phenomenon depends on a more general result. In fact, given a K3 surface S and a line bundle L on it, if there exists a smooth curve $C \in |L|$ of minimal gonality k such that the variety $W_k^1(C)$ has a reduced and isolated point, then all smooth curves in $|L|$ have gonality k , as the following argument implies.

Let A be a g_k^1 on C corresponding to a reduced and isolated point of $W_d^1(C)$ and $E := E_{C,A}$ be the Lazarsfeld-Mukai bundle associated with it. We denote by U the open subscheme of $G(2, H^0(S, E))$ consisting of vector subspaces Λ such that the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \rightarrow E$ drops rank along a smooth, irreducible curve $C_\Lambda \in |L|$. The tangent space of U at Λ is canonically identified with $H^0(C_\Lambda, A_\Lambda) \otimes H^0(C_\Lambda, \omega_{C_\Lambda} \otimes A_\Lambda^\vee)$, where $\omega_{C_\Lambda} \otimes A_\Lambda^\vee$ is the cokernel of ev_Λ . One can easily check that the derivative of the natural map $\chi_E : U \rightarrow |L|_s$ at Λ coincides with the Petri map μ_{0,A_Λ} . As a consequence, one finds that

$$\dim \chi_E(U) \geq 2(g-d+1) - \dim \ker \mu_{0,A} = \dim \text{Im} \mu_{0,A}.$$

1.6 Gonality and Clifford dimension of K3-sections

Since $A \in W_k^1(C)$ is reduced and isolated, then

$$\dim T_A W_k^1(C) = g - \dim \operatorname{Im} \mu_{0,A} = 0,$$

hence χ_E is dominant. One concludes that the projection $\pi : \mathcal{W}_k^1(|L|) \rightarrow |L|_S$ is surjective because its image is closed in $|L|_S$.

This result was strengthened by Ciliberto and Pareschi (cf. [15, Theorem B]), who showed that, if $C \in |L|_S$ is a curve of minimal gonality k such that $W_k^1(C)$ has a reduced component W , then either S and L are as in Donagi and Morrison's example or the gonality of curves in $|L|_S$ is constant. As a consequence, they obtained the following.

Theorem 1.6.1. *Let S be a K3 surface and L an ample line bundle on it. If the gonality of curves in $|L|_S$ is not constant, then the pair (S, L) is as in Donagi and Morrison's example.*

The proof consists in showing that the hypothesis on the existence of a curve C lying in $|L|_S$ and a reduced irreducible component $W \subset W_k^1(C)$ is automatically satisfied as soon as L is ample. By an infinitesimal argument, it is enough to exhibit a complete and base point free $A \in W_k^1(C)$ such that the Lazarsfeld-Mukai bundle $E_{C,A}$ is rigid, that is, $H^1(S, E_{C,A} \otimes E_{C,A}^\vee) = 0$. Ciliberto and Pareschi proved that, if a pair (C', A') computes the minimal gonality k and $\rho(g, 1, k) < 0$, the Donagi-Morrison extension corresponding to the bundle $E_{C',A'}$ has the form

$$0 \rightarrow N \rightarrow E_{C',A'} \rightarrow M \rightarrow 0,$$

with N and M being base point free and non-trivial and $h^1(S, N) = h^1(S, M) = 0$. As a consequence, the bundle $N \oplus M$ coincides with the Lazarsfeld-Mukai bundle $E_{C,A}$ associated with a pair (C, A) . Up to slightly modifying N and M , one can assume that $h^1(S, E_{C,A} \otimes E_{C,A}^\vee) = h^1(S, N \otimes M^\vee) = 0$ as desired.

More recently, Knutsen (cf. [43]) showed that Theorem 1.6.1 holds even without the ampleness assumption by a degeneration argument. His idea was to deform a pair (S, L) , such that L is globally generated but non-ample and the gonality of curves in $|L|_S$ varies, in order to make L “almost ample”, that is, the morphism ϕ_L contracts a unique rational curve Γ such that $\Gamma \cdot L = 0$ and the line bundle $H := L(-\Gamma)$ is still globally generated. Since the non-constancy of the gonality is preserved by deformations, Knutsen specialized to a curve $C' \cup \Gamma$, where $C' \in |H|$.

In [15] and [43], the Clifford dimension of curves on K3 surfaces was studied as well.

Theorem 1.6.2. *Let S be a K3 surface and $C \subset S$ a smooth curve such that $\operatorname{Cliff}(C)$ equals $\operatorname{gon}(C) + 3$. Then, one of the following occurs:*

(i) *S and $L := \mathcal{O}_S(C)$ are as in Donagi and Morrison's example.*

(ii) *$C \equiv 2D + \Gamma$, where D and Γ are smooth curves such that $D^2 \geq 2$, Γ is rational and $D \cdot \Gamma = 1$; furthermore, there is no line bundle B on S such that $0 \leq B^2 < D^2$ and $0 < B \cdot L - B^2 \leq D^2$.*

1 Introduction

The proof relies on Green and Lazarsfeld's result concerning the constancy of the Clifford index. Notice that in case (ii), if D has genus r , the genus of any curve $C \in |L|_s$ equals $4r - 2$ and $\text{Cliff}(C) = 2r - 3 = \text{gon}(C) - 3$; the only line bundle computing $\text{Cliff}(C)$ is $\mathcal{O}_C(D)$, hence C has Clifford dimension r . The same examples with $\text{Pic}(S) = \mathbb{Z}D \oplus \mathbb{Z}\Gamma$ had already been displayed by Eisenbud, Lange, Martens and Schreyer in [25], where it was conjectured that any smooth curve of Clifford dimension $r \geq 3$ has genus r and Clifford index $2r - 3$. Ciliberto and Knutsen's results imply that this conjecture holds for curves on $K3$ surfaces.

1.7 Green's Conjecture for curves on $K3$ surfaces

As briefly mentioned in Section 1.5, the constancy of Clifford index for algebraically equivalent $K3$ -sections was conjectured by Green in [33] as a direct consequence of the following conjecture concerning syzygies of canonical curves. Recall that, if $[C] \in M_g$ and $A \in \text{Pic}(C)$, the Koszul group $K_{p,q}(C, A)$ is defined as the cohomology group of the complex

$$\bigwedge^{p+1} H^0(A) \otimes H^0(A^{q-1}) \rightarrow \bigwedge^p H^0(A) \otimes H^0(A^q) \rightarrow \bigwedge^{p-1} H^0(A) \otimes H^0(A^{q+1}),$$

where all the global sections are taken over C . If A is very ample, the Koszul cohomology of A is strictly connected with the syzygies of the embedded curve

$$\phi_A : C \rightarrow \mathbb{P}(H^0(C, A)^\vee).$$

For instance, A is normally generated whenever $K_{0,q}(C, A) = 0$ for all $q \geq 2$. Analogously, the vanishing of the groups $K_{1,q}(C, A)$ for $q \geq 2$ is equivalent to the ideal of $\phi_A(C)$ being generated by quadrics.

Green's Conjecture. $K_{p,2}(C, \omega_C) = 0$ if $p < \text{Cliff}(C)$.

If true, such a statement would be optimal; indeed, Green and Lazarsfeld proved (cf. [33, Appendix]) that $K_{p,2}(C, \omega_C) \neq 0$ if $p \geq \text{Cliff}(C)$, by producing a non-zero syzygy from any linear series on C which contributes to the Clifford index. Quite remarkably, Green's Conjecture predicts that the Clifford index of C can be read off the equations of its canonical embedding. For instance, one obtains that $C \subset \mathbb{P}^{g-1}$ is projectively normal whenever $\text{Cliff}(C) > 0$, or equivalently, C is non-hyperelliptic; this is precisely the Max Noether Theorem (cf. [5, III.2]). Furthermore, a canonical curve $C \subset \mathbb{P}^{g-1}$ is cut out by quadrics as soon as $\text{Cliff}(C) > 1$; since the Clifford index of a smooth non-hyperelliptic curve C equals 1 whenever C is either trigonal or a plane quintic, one recovers the Enriques-Babbage Theorem.

If C lies on a $K3$ surface and $L := \mathcal{O}_S(C)$ is ample, the adjunction formula and Green's hyperplane section theorem (cf. [33, Theorem (3.b.7)]) imply that

$$K_{p,q}(C, \omega_C) \simeq K_{p,q}(S, L);$$

hence, smooth curves in $|L|$ form a family of canonical curves with constant syzygies and, if Green's Conjecture holds, they all have the same Clifford index.

Curves lying on K3 surfaces were used by Voisin in order to prove Green's Conjecture for general curves in any gonality stratum $M_{g,k}^1$ with $k \geq g/3 + 1$; observe that the cases of smaller gonality had earlier been worked out by Teixidor (cf. [64]) by adapting the theory of limit linear series to vector bundles of higher rank. By semicontinuity and the irreducibility of the loci $M_{g,k}^1$, it is enough to find a single curve C of gonality k which satisfies $K_{2,k-3}(C, \omega_C) = 0$; indeed, the Clifford index of general curves in $M_{g,k}^1$ is $k - 2$.

In [65], Voisin proved the following:

Theorem 1.7.1. *Let S be a K3 surface such that $\text{Pic}(S) = \mathbb{Z} \cdot L$ and $L^2 = 2g - 2$ with g even. Then smooth curves in the linear system $|L|$ satisfy Green's Conjecture.*

Notice that, under the above hypotheses, Lazarsfeld's Theorem forces every curve in $|L|_s$ to have maximal gonality $k = (g + 2)/2$. However, this result implies the generic Green's Conjecture not only when the genus is even and the gonality is maximal, but in all the cases where $g/3 + 1 \leq k \leq g/2 + 1$ by degenerating curves in $|L|_s$ to nodal curves X with at most $(k - 1)/2$ nodes.

The remaining case of odd genus $g = 2k - 3$ and maximal gonality k is performed in [66] by considering a K3 surface S such that $\text{Pic}(S) = \mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot E$, where L is very ample with $L^2 = 2g - 2$ and $g = 2k - 3$, while $E = \mathcal{O}_S(\Gamma)$ for a rational curve Γ and $E \cdot L = 2$. It turns out that, in order to prove Green's Conjecture for curves in $|L|$, it is enough to show that it holds for curves in $|L \otimes E|$ and this is obtained by slightly modifying the arguments of [65] so as to deal with the larger Picard group of S .

Green's Conjecture for general curves of odd genus $g = 2k - 3$, together with a result of Hirschowitz and Ramanan (cf. [37]), implies that the locus $Z_{g,k-3}$, consisting of curves $[C] \in M_g$ such that $K_{2,k-3}(C, \omega_C) \neq 0$, is a divisor in M_g and coincides set theoretically with the Brill-Noether divisor $M_{g,k-1}^1$. In particular, any curve of odd genus and maximal gonality satisfies Green's Conjecture. By generalizing this result to stable singular curves, in [2] Aprodu provided a sufficient condition for a curve C of genus g and gonality $k \leq (g + 2)/2$ to satisfy Green's Conjecture only in terms of the Brill-Noether theory of C , namely the *linear growth condition*

$$W_{k+n}^1(C) \leq n \text{ for } 0 \leq n \leq g - 2k + 2. \quad (1.7)$$

This is achieved by considering a nodal curve $X \in \overline{M}_{2g+3-2k}$ obtained from C by identifying $g + 3 - 2k$ pairs of general points.

The following more recent result is due to Aprodu and Farkas (cf. [3]).

Theorem 1.7.2. *Green's Conjecture holds true for every smooth curve C lying on an arbitrary K3 surface S .*

In fact, if C is general in its linear system $|L|$, has Clifford dimension 1 and gonality $k \leq (g + 2)/2$, then C satisfies condition (1.7). On the other hand, if C has higher Clifford dimension, then it does not satisfy the linear growth condition and Green's

1 Introduction

Conjecture is proved by using Knutsen's Theorem together with a degeneration argument. Now, I recall how to verify condition (1.7) in the first case.

Let $d := k + n$ and consider the projection $\pi : \mathcal{W}_d^1(|L|) \rightarrow |L|_s$. One has to show that, if curves in $|L|_s$ have genus g and gonality k and $0 \leq n \leq g - 2k + 2$, every dominating component $\mathcal{W} \subset \mathcal{W}_d^1(|L|)$ has dimension at most $g + n$. By Proposition 1.4.2, if $C \in |L|_s$ is general and $A \in W_d^1(C)$, the simplicity of $E_{C,A}$ implies that $W_d^1(C)$ is smooth in A of dimension $\rho(g, 1, d) \leq n$. As a consequence, it is enough to estimate the dimension of the irreducible components of $\mathcal{W}_d^1(|L|)$ dominating $|L|$, whose general points correspond to non-simple Lazarsfeld-Mukai bundles. For this purpose, the following parameter count is performed.

Fix a positive integer l and a non-trivial globally generated line bundle N such that, having set $M := L \otimes N^\vee$, one has $H^0(S, M \otimes N^\vee) \neq 0$. The space $P_{N,l}$, parametrizing Lazarsfeld-Mukai bundles E that are given by a non-trivial Donagi-Morrison extension

$$0 \rightarrow M \rightarrow E \rightarrow N \otimes I_\xi \rightarrow 0, \quad (1.8)$$

with $l(\xi) = l$, turns out to be an open subset of a projective bundle $\tilde{P}_{N,l}$ on the Hilbert scheme $S^{[l]}$ of 0-dimensional subschemes of S of length l . Indeed, the fibre of $\tilde{P}_{N,l}$ over $\xi \in S^{[l]}$ is $\mathbb{P}\text{Ext}^1(N \otimes I_\xi, M)$ and, surprisingly enough, its dimension does not depend on ξ .

When $P_{N,l}$ is non-empty, let $\mathcal{G}_{N,l}$ be the Grassmann bundle whose fibre over a point $[E] \in P_{N,l}$ coincides with $G(2, H^0(S, E))$. For $d := c_2(E) = c_1(M) \cdot c_1(N) + l$, one defines a rational map

$$h_{N,l} : \mathcal{G}_{N,l} \dashrightarrow \mathcal{W}_d^1(|L|),$$

by sending a general pair (E, Λ) to the point (C_Λ, A_Λ) , where C_Λ is the degeneracy locus of the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \rightarrow E$ and $\omega_{C_\Lambda} \otimes A_\Lambda^\vee$ is its cokernel. The fibre of $h_{N,l}$ over a pair (C, A) is isomorphic to $\mathbb{P}\text{Hom}(E_{C,A}, \omega_C \otimes A^\vee)$, whose dimension equals $h^0(S, E_{C,A} \otimes E_{C,A}^\vee) - 1$ by (1.4). Since (1.8) does not split by assumption, one computes that $h^0(S, E_{C,A} \otimes E_{C,A}^\vee) = 1 + h^0(S, N^\vee \otimes M)$ and, putting all together, the closure \mathcal{W} of the image of $h_{N,l}$ satisfies

$$\dim \mathcal{W} = g + l = g + d - c_1(M) \cdot c_1(N).$$

The hypothesis on the Clifford dimension of a general curve $C \in |L|_s$ is used in order to bound from below the intersection product $c_1(M) \cdot c_1(N)$. Indeed, since both $h^0(S, N) \geq 2$ and $h^0(S, M) \geq 2$, the restriction of M to C contributes to the Clifford index and the inequality

$$\text{Cliff}(M \otimes \mathcal{O}_C) \geq \text{Cliff}(C) = k - 2$$

gives $c_1(M) \cdot c_1(N) \geq k$, hence $\dim \mathcal{W} \leq g + n$, where $n := d - k$.

Concerning decomposable Lazarsfeld-Mukai bundles $E = M \oplus N$, if the rational map $\chi_E : G(2, H^0(S, E)) \dashrightarrow |L|$ is dominant, its differential at a general point Λ , which coincides with the Petri map μ_{0, A_Λ} , is surjective. As a consequence, $\rho(g, 1, d) \leq 0$ and,

for a general $C \in |L|_s$, the fibre $\chi_E^{-1}(C)$ consists of isolated points of $W_d^1(C)$.

For $k \leq d \leq g - k + 2$ and $k \leq (g + 2)/2$, thanks to Aprodu's result, one obtains that smooth curves in $|L|$ satisfy Green's Conjecture. On the other hand, when d exceeds $g - k + 2$, the same parameter count shows that every dominating component \mathcal{W} of $\mathcal{W}_d^1(|L|)$ corresponds to simple Lazarsfeld-Mukai bundles. As a consequence, in the case of maximal gonality, one finds:

Theorem 1.7.3. *Let S be a K3 surface and L a globally generated line bundle on S . Assume that general curves in $|L|$ have Clifford dimension 1, genus g and possess maximal gonality $k = \lfloor (g + 3)/2 \rfloor$. Then, if $C \in |L|_s$ is general and $\rho(g, 1, d) > 0$, the variety $W_d^1(C)$ is reduced of the expected dimension $\rho(g, 1, d)$. Moreover, if g is even, the variety $W_k^1(C)$ is zero-dimensional but not necessarily reduced.*

This result gives a partial answer to the question how special from a Gieseker-Petri point of view curves lying on an arbitrary K3 surface S are. Such a problem arises naturally from Lazarsfeld's Theorem concerning the case $\text{Pic}(S) = \mathbb{Z} \cdot L$.

1.8 Moduli spaces of sheaves on K3 surfaces

Because of the relation between the Brill-Noether theory of curves on a K3 surface S and vector bundles on S , I will recall some results on moduli spaces of sheaves of fixed rank and Chern classes on projective surfaces. A generalization of the construction of the Picard scheme to the cases of vector bundles of higher rank $r > 1$ on projective surfaces is not trivial. First of all, while looking for a complete parameter space, one cannot restrict to the class of vector bundles but has to consider also torsion free sheaves which are not locally free. Secondly, as pointed out by Geometric Invariant Theory, in order to investigate the existence of a moduli space, a notion of stability must be introduced. Let S be a projective surface and H be a polarization on S . If E is a torsion free sheaf on S , the slope of E with respect to H is, by definition,

$$\mu_H(E) := \frac{c_1(E) \cdot c_1(H)}{\text{rk } E}.$$

Following Mumford and Takemoto (cf. [62]), one says that E is μ_H -stable if for every subsheaf $0 \neq F \subset E$ such that $\text{rk } F < \text{rk } E$ one has $\mu_H(F) < \mu_H(E)$, and that E is μ_H -semistable if equalities are also allowed.

For constructing compact moduli spaces, the following different notion of stability, which was introduced by Gieseker (cf. [30]), proved more useful. A torsion free sheaf E on S is said to be H -semistable if every subsheaf $F \neq 0$ of E of smaller rank satisfies $p(F, m) \leq p(E, m)$, where

$$p(E, m) := \frac{\chi(E \otimes H^{\otimes m})}{\text{rk } E}$$

is the normalized Hilbert polynomial; H -stability requires that all the inequalities are

1 Introduction

strict. One can check that

$$\mu_H - \text{stability} \Rightarrow H - \text{stability} \Rightarrow H - \text{semistability} \Rightarrow \mu_H - \text{semistability};$$

moreover, every H -stable sheaf is simple.

In [30], Gieseker first constructed the moduli space $M_H(\underline{c})$ of semistable sheaves of fixed Chern classes \underline{c} on (S, H) . The space $M_H(\underline{c})$ is a projective variety parametrizing S -equivalence classes of H -semistable objects. I recall that every H -semistable sheaf E admits a filtration, the so-called Jordan-Hölder filtration, which splits E in stable factors $gr_i(E)$ and the graded object $gr(E) := \oplus_i gr_i(E)$ is uniquely determined. Two semistable sheaves are S -equivalent whenever they have the same graded object. This notion was originally introduced by Seshadri (cf. [58]), who explained that, in order to obtain a separated moduli space for semistable vector bundles of rank $r > 1$ on an algebraic curve, one has to consider them up to S -equivalence instead then up to isomorphism. The space $M_H(\underline{c})^s$, parametrizing isomorphism classes of H -stable sheaves, is an open and dense subscheme of $M_H(\underline{c})$. Since H -stability is weaker than μ_H -stability, one finds an open subset $M_H(\underline{c})^{\mu s}$ of $M_H(\underline{c})$, whose point correspond to bundles which are μ_H -stable. In Chapter 3, we will denote by $\mathcal{M}_H(\underline{c})^{\mu s}$ the Artin stack obtained from the scheme $M_H(\underline{c})^{\mu s}$ by endowing every point with the automorphism group of the corresponding μ_H -stable sheaf, which has dimension 1.

I briefly recall the idea of a later construction of $M_H(\underline{c})$ presented by Simpson in [60]. If m is sufficiently high, to any torsion free sheaf F of Chern classes \underline{c} together with a basis for $H^0(S, F \otimes H^{\otimes m})$, one associates a surjection

$$E := (H^{\otimes -m})^{\oplus n} \rightarrow F,$$

where $n := h^0(S, F \otimes H^{\otimes m})$. Since such a morphism naturally defines a point of the Grothendieck's Quot scheme $Quot(E, P)$, parametrizing quotients of E with the same Hilbert polynomial P as F , the space $M_H(\underline{c})$ can be constructed starting from an open subset of $Quot(E, P)$ and modding it out by the freedom in the choice of the basis for $H^0(S, F \otimes H^{\otimes m})$.

The moduli spaces $Spl(\underline{c})$ of simple sheaves on S of Chern classes $\underline{c} = (\text{rk}, c_1, c_2)$ was constructed in the complex case by Kosarew and Okonek in [44]. The tangent space of $Spl(\underline{c})$ at a point $[E]$ is canonically isomorphic to $\text{Ext}^1(E, E)$ (cf. [1]). In [47], Mukai proved that, if S is a $K3$ surface, $Spl(\underline{c})$ is smooth of dimension $(1 - \text{rk})c_1^2 - 2(\text{rk})^2 + 2\text{rk}c_2 + 2$. This result can be used in order to simplify the proof of Lazarsfeld's Theorem in the following way (cf. [46]).

I recall that the hypothesis $\text{Pic}(S) = \mathbb{Z} \cdot L$ forces all the Lazarsfeld-Mukai bundles to be simple. In particular, if $c_1(L)^2 = 2g - 2$ and $C \in |L|_s$ has a linear series of type g_d^r , one has $\rho(g, r, d) \geq 0$. By the generic smoothness theorem applied to the projection $\pi : \mathcal{W}_d^r(|L|) \rightarrow |L|_s$, in order to prove that the Petri map associated with any complete, base point free $A \in W_d^r(C)$ is injective, it is enough to show that the variety $\mathcal{W}_d^r(|L|)$ is smooth of dimension $g + \rho(g, r, d)$. Set $\underline{c} := (r + 1, C, d)$ and let \mathcal{G} be the Grassmann bundle on $Spl(\underline{c})$ whose fibre over a point $[E]$ coincides with $G(r + 1, H^0(S, E))$. Con-

sider the open subset U of \mathcal{G} consisting of pairs (E, Λ) such that E is locally free and globally generated, $h^1(S, E) = h^2(S, E) = 0$ and the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \rightarrow E$ drops rank along a smooth curve C_Λ and has a line bundle $\omega_{C_\Lambda} \otimes A_\Lambda^\vee$ as cokernel. The fibre over (C, A) of the natural map $h : U \rightarrow W_d^r(|L|)$, sending a pair (E, Λ) to (C_Λ, A_Λ) , is isomorphic to $\mathbb{P}\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^\vee) \simeq \mathbb{P}H^0(C, E_{C,A} \otimes E_{C,A}^\vee)$, which is a point. Hence, the variety $W_d^r(|L|)$ is isomorphic to U and Mukai's Theorem leads to the conclusion.

For completion, I recall that Mukai's result is actually much stronger, also stating that $Spl(\underline{c})$ is a symplectic manifold, that is, it has a 2-form which vanishes nowhere. Since $M_H(\underline{c})^s$ is an open subset of $Spl(\underline{c})$, it is smooth of the same dimension as $Spl(\underline{c})$ and it inherits its symplectic structure. In the particular case where $M_H(\underline{c})$ is 2-dimensional, Mukai showed that it is a K3 surface isogenous to S (cf. [48]). In higher dimensional cases, under some mild hypotheses on \underline{c} and H , O'Grady (cf. [51]) proved that $M_H(\underline{c})$ is an irreducible hyperkähler manifold; moreover, $M_H(\underline{c})$ is deformation equivalent to the Hilbert scheme $S^{[n]}$ for some n by a result of Huybrechts (cf. [38]).

1.9 Outline of the results

In Chapter 2 we study the Gieseker Petri locus GP_g in low genera. It is conjectured that this has pure codimension 1 in M_g , as proved by Castorena up to genus 8 (cf. [11, 13]). We extend this result by showing that:

Theorem. *The Gieseker-Petri locus GP_g has pure codimension 1 in \mathcal{M}_g for $g \leq 13$.*

The idea is to consider all the loci $GP_{g,d}^r$, which may be themselves reducible, and to prove that the ones whose codimension is either unknown or strictly greater than 1 are contained in some divisorial components of GP_g . The cases $g = 9, 10, 11$ follow from some general inclusions together with a recent result of Bruno and Sernesi.

For $g = 12, 13$ the situation becomes more complicated. In particular, it is necessary to develop a new method in order to control the codimension of the locus $GP_{g,g-2}^1$. We remove some annoying hypotheses in a previous result of Castorena ([12]), thus exhibiting a divisorial component of $GP_{g,g-2}^1$ for all g . This consists of curves $[C]$ in $GP_{g,g-2}^1$ such that, if $L \in W_{g-2}^1(C)$ satisfies $\ker \mu_{0,L} \neq 0$, then L is complete and base point free and $\omega_C \otimes L^\vee \in W_g^2(C)$ is big.

When the genus becomes higher, the number of components of the Gieseker-Petri locus increases and it becomes more and more difficult to prove the conjectural statement that GP_g has pure codimension 1 in M_g . We strongly believe that some Brill Noether loci $\mathcal{M}_{g,d}^r$ with $\rho(g, r, d) < -1$ might be irreducible components of GP_g , thus making the conjecture fail.

As suggested by Lazarsfeld, the behavior of curves lying on a K3 surface S with respect to Petri maps is quite interesting. The Gieseker-Petri Theorem generally fails if the Picard number of S is greater than 1 and it is natural to ask up to which extent.

1 Introduction

In Chapter 3 we look at linear series of type g_d^2 on a curve C lying on an arbitrary K3-surface S . If $\rho(g, 2, d) < 0$, we show that, under some mild hypotheses on $L := \mathcal{O}_S(C)$, any g_d^2 on C is contained in the restriction to C of a line bundle on S as predicted by the Donagi and Morrison's Conjecture. More precisely, we prove the following.

Theorem. *Let S be a K3 surface and $L \in \text{Pic}(S)$ be an ample line bundle such that a general curve in $|L|$ has genus g , Clifford dimension 1 and maximal gonality $k = \lfloor \frac{g+3}{2} \rfloor$. Let A be a complete, base point free g_d^2 on a curve $C \in |L|_S$ such that $\rho(g, 2, d) < 0$.*

Then, there exists $M \in \text{Pic}(S)$ adapted to $|L|$ such that the linear system $|A|$ is contained in $|M \otimes \mathcal{O}_C|$ and $\text{Cliff}(M \otimes \mathcal{O}_C) \leq \text{Cliff}(A)$. Moreover, one has $c_1(M) \cdot C \leq (4g - 4)/3$.

If $\rho(g, 2, d) \geq 0$ and C is general in its linear system, we look at the subvariety $\tilde{W}_d^2(C)$ of $W_d^2(C)$ whose points correspond to base point free line bundles.

Theorem. *Under the same hypotheses on (S, L) as above, fix a positive integer d such that $\rho(g, 2, d) \geq 0$ and $(g, d) \notin \{(2, 4), (4, 5), (6, 6), (10, 9)\}$. Then, for a general $C \in |L|_S$, the following hold.*

- a. *If $d > \frac{3}{4}g + 2$, the variety $\tilde{W}_d^2(C)$ is reduced of the expected dimension $\rho(g, 2, d)$.*
- b. *If $d \leq \frac{3}{4}g + 2$, let W be an irreducible component of $\tilde{W}_d^2(C)$ which either is non-reduced or has dimension greater than $\rho(g, 2, d)$. Then, there exists an effective divisor $D \subset S$ such that $\mathcal{O}_S(D)$ is adapted to $|L|$ and, for a general $A \in W$, the linear system $|A|$ is contained in $|\mathcal{O}_C(D)|$ and*

$$\text{Cliff}(\mathcal{O}_C(D)) \leq \text{Cliff}(A).$$

For d large enough, this gives an analogue of Aprodu and Farkas' dimensional statement for the variety $W_d^1(C)$. If instead $d \leq \frac{3}{4}g + 2$, the theorem implies that general points of a component of $\tilde{W}_d^2(C)$, which either is non-reduced or has dimension greater than the expected one, correspond to linear series which are all contained in the restriction to C of the same line bundle on S .

The study of linear series of type g_d^2 involves vector bundles of rank 3 and one cannot control the non-simple ones by means of Donagi-Morrison extensions. Our new approach consists of studying the simplicity of rank-3 Lazarsfeld-Mukai bundles indirectly, by investigating whether they are μ_L -stable. We use the well known fact that every μ_L -unstable sheaf E has a filtration, called the Harder-Narasimhan filtration, which splits E in μ_L -semistable factors. Analogously, if E is properly μ_L -semistable, we consider its Jordan-Hölder filtration by μ_L -stable factors. We show that when d is small enough, in particular always for $\rho(g, 2, d) < 0$, the Lazarsfeld-Mukai bundle E associated with any A of type g_d^2 on any curve $C \in |L|_S$ fits into a short exact sequence

$$0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0,$$

where $N \in \text{Pic}(S)$ and E/N is a μ_L -stable, torsion free sheaf of rank 2. Such extensions will play the same role as Donagi-Morrison extensions in the case of rank 2. I reckon

that the use of Harder-Narasimhan and Jordan-Hölder filtrations is the right strategy in order to prove the Donagi and Morrison's Conjecture in general.

The dimensional statement for the variety $\tilde{W}_d^2(C)$ when $\rho(g, 2, d) \geq 0$ follows from bounding the number of moduli of μ_L -unstable and properly μ_L -semistable Lazarsfeld-Mukai bundles of rank 3 by using some Artin stacks that parametrize the corresponding Harder-Narasimhan and Jordan-Hölder filtrations, which can be of various types.

The aforementioned results are used in order to prove the transversality of some components of the Gieseker-Petri locus GP_g in any genus g . An application towards higher rank Brill-Noether theory is also presented.

In Chapter 4 we study syzygies of curves lying on Del Pezzo surfaces and obtain the following:

Theorem. *Let C be a smooth, irreducible curve lying on a Del Pezzo surface S and, having set $L := \mathcal{O}_S(C)$, assume that $L \otimes \omega_S$ is nef and big. Then, the following hold:*

- *If $\deg(S) \geq 2$, then C satisfies Green's Conjecture.*
- *If C is general in its linear system and $\text{Cliffdim}(C) = 1$, then C verifies Green-Lazarsfeld's Gonality Conjecture.*
- *If $\deg(S) = 1$, Green's Conjecture is true for a general curve in $|L|$; under the further assumption that the Clifford index of a general curve in $|L|$ is not computed by the restriction of the anticanonical bundle ω_S^\vee , Green's Conjecture holds for every smooth irreducible curve in $|L|$.*

Curves on Del Pezzo surfaces share some common behavior with K3-sections. In particular, Pareschi (cf. [52]) and Knutsen (cf. [42]) proved that the gonality and the Clifford index of a curve C on a Del Pezzo surface S only depend on the linear equivalence class of C , with very few exceptions. Moreover, the Clifford dimension of C is at most 3.

It turns out that the linear growth condition (1.7) holds for a curve C which is general in its linear system, has Clifford dimension 1 and gonality $k \leq (g + 2)/2$. We make use of some rank-2 vector bundles $E_{C,A}$ whose definition is analogous to that of Lazarsfeld-Mukai bundles for K3-surfaces. The ampleness of the anticanonical bundle ω_S^\vee assures that the injectivity of the Petri map associated with a complete, base point free pencil A on C depends on the group $H^2(S, E_{C,A} \otimes E_{C,A}^\vee)$, which is trivially zero if $E_{C,A}$ is stable. We thus perform a parameter count for pairs (C, A) such that $E_{C,A}$ is not $\mu_{L \otimes \omega_S^\vee}$ -stable.

We get Green's Conjecture for *every* smooth irreducible curve C in $|L|$, when

$$K_{g-c-1,1}(S, L \otimes \omega_S) \simeq K_{g-c-1,1}(C, \omega_C);$$

the hypotheses that $L \otimes \omega_S$ is nef and big and that $\omega_S^\vee \otimes \mathcal{O}_C$ does not compute $\text{Cliff}(C)$ if $\deg(S) = 1$ are used in order to prove the above isomorphism of Koszul cohomology groups. We remark that the cases not satisfying such assumptions coincide with the only examples in genus ≥ 4 where the Clifford index of curves in $|L|$ is not constant.

1 Introduction

Our proof does not deeply use the ampleness ω_S^\vee , but only its effectivity. Therefore, it seems likely that the same techniques might succeed in proving Green's Conjecture for curves on other types of rational anticanonical surfaces.

Chapter 2 and Chapter 3 are respectively based on the the following two papers:

- M. Lelli-Chiesa, *The Gieseker-Petri divisor in \mathcal{M}_g for $g \leq 13$* , Geom. Dedicata 158 (2012), 149-165.
- M. Lelli-Chiesa, *Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces*, preprint [ArXiv:1112.2938](https://arxiv.org/abs/1112.2938).

Bibliography

- [1] A. B. Altman and S. L. Kleiman. Compactifying the Picard scheme. *Adv. Math.*, 35:50–112, 1980.
- [2] M. Aprodu. Remarks on syzygies of d -gonal curves. *Math. Res. Lett.*, 12(2-3):387–400, 2005.
- [3] M. Aprodu and G. Farkas. Green’s conjecture for curves on arbitrary $K3$ surfaces. *Compos. Math.*, 147(3):839–851, 2011.
- [4] E. Arbarello and M. Cornalba. Su una congettura di Petri. *Comment. Math. Helv.*, 56:1–38, 1981.
- [5] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris. *Geometry of algebraic curves. Volume I.*, volume 267 of *Grundl. Math. Wiss.* . Springer Verlag, 1985.
- [6] M. F. Atiyah. Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.*, 85:181–207, 1957.
- [7] D. Babbage. A note on the quadrics through a canonical curve. *J. Lond. Math. Soc.*, 14:310–315, 1939.
- [8] S. Bloch and D. Gieseker. The positivity of the Chern classes of an ample vector bundle. *Invent. Math.*, 12:112–117, 1971.
- [9] A. Brill and M. Noether. Ueber die algebraischen funktionen und ihre anwendungen in der geometrie. *Math. Ann.*, 7:269–310, 1873.
- [10] G. Castelnuovo. Ricerche di geometria sulle curve algebriche. *Torino Atti XXIV.*, pages 346–373, 1889.
- [11] A. Castorena. Curves of genus seven that do not satisfy the Gieseker–Petri theorem. *Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8)*, 8(3):697–706, 2005.
- [12] A. Castorena. A family of plane curves with moduli $3g - 4$. *Glasg. Math. J.*, 49(3):417–422, 2007.
- [13] A. Castorena. Remarks on the Gieseker–Petri divisor in genus eight. *Rend. Circ. Mat. Palermo (2)*, 59(1):143–150, 2010.
- [14] C. Ciliberto, A. Lopez, and R. Miranda. Projective degenerations of $K3$ surfaces, Gaussian maps, and Fano threefolds. *Invent. Math.*, 114(3):641–667, 1993.

Bibliography

- [15] C. Ciliberto and G. Pareschi. Pencils of minimal degree on curves on a $K3$ surface. *J. Reine Angew. Math.*, 460:15–36, 1995.
- [16] W. K. Clifford. On the classification of loci. *Math. Phil.*, 1878.
- [17] M. Coppens and G. Martens. Secant spaces and Clifford’s theorem. *Compos. Math.*, 78(2):193–212, 1991.
- [18] P. Deligne and D. Mumford. The irreducibility of the space of curves of a given genus. *Publ. Math., Inst. Hautes Étud. Sci.*, 36:75–109, 1969.
- [19] R. Donagi and D. R. Morrison. Linear systems on $K3$ -sections. *J. Differ. Geom.*, 29(1):49–64, 1989.
- [20] D. Eisenbud and J. Harris. A simpler proof of the Gieseker-Petri theorem on special divisors. *Invent. Math.*, 74:269–280, 1983.
- [21] D. Eisenbud and J. Harris. Divisors on general curves and cuspidal rational curves. *Invent. Math.*, 74:371–418, 1983.
- [22] D. Eisenbud and J. Harris. Limit linear series: Basic theory. *Invent. Math.*, 85:337–371, 1986.
- [23] D. Eisenbud and J. Harris. The Kodaira dimension of the moduli space of curves of genus ≥ 23 . *Invent. Math.*, 90:359–387, 1987.
- [24] D. Eisenbud and J. Harris. Irreducibility of some families of linear series with Brill-Noether number -1 . *Ann. Sci. Éc. Norm. Supér. (4)*, 22(1):33–53, 1989.
- [25] D. Eisenbud, H. Lange, G. Martens, and F.-O. Schreyer. The Clifford dimension of a projective curve. *Compos. Math.*, 72(2):173–204, 1989.
- [26] F. Enriques. Sulle curve canoniche di genere p nello spazio a $p - 1$ dimensioni. *Rend. Accad. Sci. Ist. Bologna.*, 83:80–82, 1919.
- [27] G. Farkas. Gaussian maps, Gieseker-Petri loci and large theta-characteristics. *J. Reine Angew. Math.*, 581:151–173, 2005.
- [28] G. Farkas. Rational maps between moduli spaces of curves and Gieseker-Petri divisors. *J. Algebr. Geom.*, 19(2):243–284, 2010.
- [29] W. Fulton and R. Lazarsfeld. On the connectedness of degeneracy loci and special divisors. *Acta Math.*, 146:271–283, 1981.
- [30] D. Gieseker. On the moduli of vector bundles on an algebraic surface. *Ann. Math. (2)*, 106:45–60, 1977.
- [31] D. Gieseker. Stable curves and special divisors: Petri’s conjecture. *Invent. Math.*, 66:251–275, 1982.

- [32] M. Green and R. Lazarsfeld. Special divisors on curves on a K3 surface. *Invent. Math.*, 89:357–370, 1987.
- [33] M. L. Green. Koszul cohomology and the geometry of projective varieties. *J. Differ. Geom.*, 19:125–171, 1984.
- [34] P. Griffiths and J. Harris. The dimension of the variety of special linear systems on a general curve. *Duke Math. J.*, 47:233–272, 1980.
- [35] J. Harris. On the Kodaira dimension of the moduli space of curves. II: The even-genus case. *Invent. Math.*, 75:437–466, 1984.
- [36] J. Harris and D. Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67:23–86, 1982.
- [37] A. Hirschowitz and S. Ramanan. New evidence for Green’s conjecture on syzygies of canonical curves. *Ann. Sci. Éc. Norm. Supér. (4)*, 31(2):145–152, 1998.
- [38] D. Huybrechts. Birational symplectic manifolds and their deformations. *J. Differ. Geom.*, 45(3):488–513, 1997.
- [39] G. Kempf. *Schubert methods with an application to algebraic curves*. Amsterdam, 1971.
- [40] S. Kleiman and D. Laksov. On the existence of special divisors. *Am. J. Math.*, 94:431–436, 1972.
- [41] F. Klein. *Über Riemann’s Theorie der algebraischen Functionen und ihrer Integrale*. Leipzig, Teubner, 1882.
- [42] A. L. Knutsen. Exceptional curves on Del Pezzo surfaces. *Math. Nachr.*, 256:58–81, 2003.
- [43] A. L. Knutsen. On two conjectures for curves on K3 surfaces. *Int. J. Math.*, 20(12):1547–1560, 2009.
- [44] S. Kosarew and C. Okonek. Global moduli spaces and simple holomorphic bundles. *Publ. Res. Inst. Math. Sci.*, 25(1):1–19, 1989.
- [45] R. Lazarsfeld. Brill-Noether-Petri without degenerations. *J. Differ. Geom.*, 23:299–307, 1986.
- [46] R. Lazarsfeld. A sampling of vector bundle techniques in the study of linear series. In *Lectures on Riemann surfaces (Trieste, 1987)*, pages 500–559. World Sci. Publ., Teaneck, NJ, 1989.
- [47] S. Mukai. Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. *Invent. Math.*, 77:101–116, 1984.

Bibliography

- [48] S. Mukai. On the moduli space of bundles on $K3$ surfaces. I. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 341–413. Tata Inst. Fund. Res., Bombay, 1987.
- [49] D. Mumford. Stability of projective varieties. *Enseignement Math.*, 23(1-2):39–110, 1977.
- [50] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*. 3rd enl. ed. Berlin: Springer-Verlag, 1993.
- [51] K. G. O’Grady. The weight-two Hodge structure of moduli spaces of sheaves on a K^3 surface. *J. Algebr. Geom.*, 6(4):599–644, 1997.
- [52] G. Pareschi. Exceptional linear systems on curves on Del Pezzo surfaces. *Math. Ann.*, 291(1):17–38, 1991.
- [53] G. Pareschi. A proof of Lazarsfeld’s theorem on curves on $K3$ surfaces. *J. Algebr. Geom.*, 4(1):195–200, 1995.
- [54] K. Petri. Über Spezialkurven. I. *Math. Ann.*, 93:182–209, 1925.
- [55] M. Reid. Special linear systems on curves lying on a $K3$ surface. *J. Lond. Math. Soc., II. Ser.*, 13:454–458, 1976.
- [56] B. Riemann. Theorie der Abel’schen Functionen. *J. Reine Angew. Math.*, 54:115–155, 1857.
- [57] B. Saint-Donat. Projective models of $K3$ surfaces. *Am. J. Math.*, 96:602–639, 1974.
- [58] C. Seshadri. Space of unitary vector bundles on a compact Riemann surface. *Ann. Math. (2)*, 85:303–336, 1967.
- [59] F. Severi. *Vorlesungen über algebraische Geometrie*. Teubner, Leipzig, 1921.
- [60] C. T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. *Publ. Math., Inst. Hautes Étud. Sci.*, 79:47–129, 1994.
- [61] F. Steffen. A generalized principal ideal theorem with an application to Brill-Noether theory. *Invent. Math.*, 132(1):73–89, 1998.
- [62] F. Takemoto. Stable vector bundles on algebraic surfaces. *Nagoya Math. J.*, 47:29–48, 1972.
- [63] Teichmüller, Oswald. *Gesammelte Abhandlungen. Collected papers*. Hrsg. von L. V. Ahlfors und F. W. Gehring. Berlin, New York: Springer-Verlag, 1982.
- [64] M. Teixidor i Bigas. Green’s conjecture for the generic r -gonal curve of genus $g \geq 3r - 7$. *Duke Math. J.*, 111(2):195–222, 2002.

- [65] C. Voisin. Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface. *J. Eur. Math. Soc. (JEMS)*, 4(4):363–404, 2002.
- [66] C. Voisin. Green's canonical syzygy conjecture for generic curves of odd genus. *Compos. Math.*, 141(5):1163–1190, 2005.

2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$

2.1 Introduction

Let M_g be the coarse moduli space of smooth irreducible projective curves of genus g . Given $[C] \in M_g$ and a line bundle A on C , we consider the Petri map

$$\mu_{0,A} : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee) \rightarrow H^0(C, \omega_C).$$

This map has been studied in detail because of its importance in the description of the Brill-Noether varieties $G_d^r(C)$ and $W_d^r(C)$. The most important result in this sense is the Gieseker-Petri Theorem (cf. [16], [9]), which asserts that for the generic curve and for any line bundle on it the Petri map is injective. This implies that if $[C] \in M_g$ is general and the Brill-Noether number

$$\rho(g, r, d) := g - (r + 1)(g - d + r)$$

is nonnegative, then $G_d^r(C)$ is smooth of dimension $\rho(g, r, d)$ and the natural morphism $G_d^r(C) \rightarrow W_d^r(C)$ is a rational resolution of singularities. We refer to Section 1.2 for details. The Gieseker-Petri locus is defined as

$$GP_g := \{[C] \in M_g \mid C \text{ does not satisfy the Gieseker-Petri Theorem}\}.$$

It is conjectured that GP_g has pure codimension 1 inside M_g ; an explanation why this is plausible is given below. The expectation has been proved in genus up to 8 by Castorena (cf. [5], [7]). Our main result is:

Theorem 2.1.1. *The Gieseker-Petri locus GP_g has pure codimension 1 inside M_g for $g \leq 13$.*

Our strategy is to look at the different components of GP_g determined by the numerical type of the linear series for which the Gieseker-Petri Theorem fails. For values of g, r, d such that both $r + 1$ and $g - d + r$ are at least 2 we consider the Gieseker-Petri locus of type (r, d)

$$GP_{g,d}^r := \{[C] \in M_g \mid \exists (A, V) \in G_d^r(C) \text{ with } \ker \mu_{0,V} \neq 0\},$$

where $\mu_{0,V}$ denotes the restriction of the Petri map to $V \otimes H^0(C, \omega_C \otimes A^\vee)$. Clifford's Theorem, along with Riemann-Roch Theorem, restricts to the range $0 < 2r \leq d \leq g - 1$ the values of g, r, d for which it is necessary to study the component $GP_{g,d}^r$. We also

2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$

recall that, given $[C] \in GP_g$, at least one of the linear series on C for which the Gieseker-Petri Theorem fails is *primitive*, that is, complete and such that both A and $\omega_C \otimes A^\vee$ are base point free.

In some cases the codimension of $GP_{g,d}^r$ inside M_g is known but in general it seems quite difficult to determine the irreducible components of GP_g and control their dimension. When $\rho(g, r, d) < 0$, the Petri map corresponding to any g_d^r on a genus g curve cannot be injective for dimension reasons and the locus $GP_{g,d}^r$ coincides with the Brill-Noether variety

$$M_{g,d}^r := \{[C] \in M_g \mid W_d^r(C) \neq \emptyset\}.$$

In particular, when $\rho(g, r, d) = -1$, the locus $M_{g,d}^r$, if nonempty, is an irreducible divisor (cf. [11], [22]), known as the Brill-Noether divisor. On the other side, if $\rho(g, r, d) < -1$, the codimension of any component Z of $M_{g,d}^r$ in M_g is strictly greater than 1. If it is true that GP_g has pure codimension 1 inside M_g , then Z must be contained in some *divisorial*¹ component of GP_g ; there is no evident reason why this should hold in general.

When $\rho(g, r, d) \geq 0$, the Gieseker-Petri locus $GP_{g,d}^r$ can be described as the image of a determinantal variety under a projection map. Indeed, given a family $\varphi : \mathcal{C} \rightarrow S$ of smooth curves of genus g admitting a section, one can define a variety $\pi : \mathcal{G}_{\mathcal{C}/S}^{r,d} \rightarrow S$ parametrizing pairs $(C_s, (A_s, V_s))$ such that $C_s = \varphi^{-1}(s)$ for some $s \in S$ and (A_s, V_s) in $G_d^r(C_s)$. On $\mathcal{G}_{\mathcal{C}/S}^{r,d}$, there exists a morphism of vector bundles

$$\mu : \mathcal{E}_1 \otimes \mathcal{E}_2 \rightarrow \mathcal{F},$$

which extends the Petri map. The codimension in $\mathcal{G}_{\mathcal{C}/S}^{r,d}$ of the degeneracy locus $X(\mu)$ is at most $\rho(g, r, d) + 1$. The locus

$$GP_{\mathcal{C}/S}^{r,d} := \{s \in S \mid [C_s] \in GP_{g,d}^r\}$$

coincides with the image under π of $X(\mu)$; the transversality of the fibres of π to $X(\mu)$ would imply that $GP_{\mathcal{C}/S}^{r,d}$ has codimension 1 in S . Since the construction can be globalized by considering the universal family on the moduli stack \mathcal{M}_g , this suggests that the locus $GP_{g,d}^r$ is divisorial.

We define

$${}^bGP_{g,d}^r := \{[C] \in M_g \mid \text{a base point free } (A, V) \in G_d^r(C) \text{ with } \ker \mu_{0,V} \neq 0\};$$

this locus is open in $GP_{g,d}^r$ but not necessarily dense.

Farkas proved that ${}^b\overline{GP}_{g,d}^r$ always has a divisorial component if $\rho(g, r, d) \geq 0$ (cf. [13], [14]). However, there are only few cases when $GP_{g,d}^r$ is well understood. In Section 2.2, we will recall known results on some irreducible components of GP_g of type both $GP_{g,d}^r$ with $\rho(g, r, d) \geq 0$ and $M_{g,d}^r$ for $\rho(g, r, d) < 0$.

We summarize our results. We show that when $g \leq 13$ the components of GP_g whose

¹By divisorial we will always mean a locus of pure codimension 1.

codimension is either unknown or strictly greater than 1 are contained in some divisorial components. Most of the inclusions easily follow from some basic facts established in Section 2.3. In particular, the components $GP_{g,k}^1$ with $\rho(g, 1, k) < -1$ are all contained in the Brill-Noether divisor $M_{g, \frac{g+1}{2}}^1$ if g is odd, and in the locus $GP_{g, \frac{g+2}{2}}^1$ if g is even.

If the Brill-Noether number $\rho(g, r, d)$ is either 0 or 1, we can prove the inclusion of both $M_{g,d+1}^{r+1}$ and $M_{g,d-1}^r$ inside $GP_{g,d}^r$. We use a very recent result, due to Bruno and Sernesi, stating that for values of g, r, d such that $\rho(g, r, d) \geq 0$ and $\rho(g, r+1, d) < 0$, the locus $GP_{g,d}^r$ is divisorial outside its intersection with $M_{g,d}^{r+1}$ (cf. [4]). As a corollary we obtain that, in even genus, $GP_{g, \frac{g+2}{2}}^1$ coincides with the closure of the locus ${}^bGP_{g, \frac{g+2}{2}}^1$, which was studied by Eisenbud and Harris.

In Section 2.4 we prove Theorem 2.1.1 in genera 9, 10, 11. In addition to the results of the previous section, we use some well known facts about plane curves. The study of the component $M_{10,9}^3$ requires extra work: we prove that it is contained in $GP_{10,6}^1$ by remarking that any curve of degree 9 and genus 10 in \mathbb{P}^3 is either a curve of type (3, 6) on a non singular quadric surface or the intersection of two cubic surfaces; linear series on a cubic surface X can be easily written down remembering that X is isomorphic to the blow-up of the projective plane in 6 points.

In the last section we deal with genera 12 and 13. The situation gets more complicated because the methods used before do not enable us to control the codimension of $GP_{g,g-2}^1$. We prove the following theorem:

Theorem 2.1.2. *Let $[C] \in GP_{g,g-2}^1$ be a non hyperelliptic curve with no vanishing theta-null. Let us assume that for any $A \in G_{g-2}^1(C)$ such that $\mu_{0,A}$ is not injective, A is primitive and $\omega_C \otimes A^\vee \in W_g^2(C)$ defines a birational morphism. Then C carries only a finite number of $A \in W_{g-2}^1(C)$ for which $\ker \mu_{0,A} \neq 0$.*

This generalizes [6], where it is assumed that the plane model Γ of C corresponding to $\omega_C \otimes A^\vee$ has only singularities which become nodes after a finite number of blow-ups (in a somewhat oldfashioned way these are called possibly infinitely near nodes). The idea of our proof is to show that we do not need any assumption on the singularities of Γ because the non injectivity of $\mu_{0,A}$ implies that Γ has at least one double point, which cannot be a cusp of any order if $[C] \notin GP_{g,g-1}^1$; then we proceed like in [6]. Theorem 2.1.2 implies Theorem 2.1.1 in genus 13 because no g_{13}^2 can be composed with an involution. Instead, for a curve $[C] \in GP_{12,10}^1$ it may happen that a g_{12}^2 , for which the Petri map is not injective, induces a finite covering of a plane curve of lower genus. We prove that this can be the case only for $[C] \in GP_{12,7}^1 \cup GP_{12,8}^1$ (cf. Theorem 2.5.5).

2.2 Divisorial components of GP_g

In this section we will recall some results concerning the codimension inside M_g of the Brill-Noether loci $M_{g,d}^r$ and of some components of the Gieseker-Petri locus whose corresponding Brill-Noether number is nonnegative.

2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$

The following theorem is due to Steffen (cf. [22]).

Theorem 2.2.1. *The codimension of the Brill-Noether locus $M_{g,d}^r$ inside M_g is $\leq -\rho(g, r, d)$. Equality holds in the cases where $\rho(g, r, d) \in \{-1, -2, -3\}$.*

The last part of the statement follows directly from the results of Eisenbud and Harris (cf. [11]) and Edidin (cf. [8]). Steffen's argument is as follows.

Let $\varphi : \mathcal{C} \rightarrow S$ be a family of smooth curves of genus g admitting a section. Having fixed an integer m such that $m + d \geq 2g - 1$, denote by $\text{Pic}_{\mathcal{C}/S}^{m+d}$ the relative Picard scheme. The construction of $W_d^r(C)$ as a determinantal variety can be extended in order to define a scheme $\mathcal{W}_{\mathcal{C}/S}^{r,d} \rightarrow S$, whose fibre over a point $s \in S$ coincides with the scheme $W_d^r(C_s)$, where $C_s = \varphi^{-1}(s)$. In fact, $\mathcal{W}_{\mathcal{C}/S}^{r,d}$ is the $(m + d - g - r)$ -th degeneracy locus of a morphism of vector bundles $\mathcal{E} \rightarrow \mathcal{F}$ on $\text{Pic}_{\mathcal{C}/S}^{m+d}$ such that \mathcal{E}, \mathcal{F} have rank $m + d - g + 1$ and m respectively. Thus, $\mathcal{W}_{\mathcal{C}/S}^{r,d}$ has codimension at most $(r + 1)(d - g + r)$ in $\text{Pic}_{\mathcal{C}/S}^{m+d}$. Moreover, one shows that the vector bundle $\mathcal{E}^\vee \otimes \mathcal{F}$ is ample relative to the projection $p : \text{Pic}_{\mathcal{C}/S}^{m+d} \rightarrow S$, that is, it is ample when restricted to any fibre of p . A more general theorem of Steffen implies that this forces $\mathcal{W}_{\mathcal{C}/S}^{r,d}$ to be transversal to the fibres of p , hence the image $p(\mathcal{W}_{\mathcal{C}/S}^{r,d})$ has codimension at most $-\rho(g, r, d)$ in S . By considering the universal family on the moduli stack \mathcal{M}_g , one globalizes the construction and obtains the statement.

When $r = 1$, a stronger result holds.

Theorem 2.2.2. *If $d < (g + 2)/2$, the Brill-Noether locus $M_{g,d}^1$ is irreducible and has codimension $-\rho(g, 1, d)$ in M_g .*

The irreducibility statement is due to Fulton (cf. [15]) and follows from the irreducibility of Hurwitz schemes together with the Riemann's Existence Theorem. Concerning the dimensional statement, given a family of curves $\varphi : \mathcal{C} \rightarrow S$, consider the projection $p : \mathcal{W}_{\mathcal{C}/S}^{1,d} \rightarrow S$. As proved by Arbarello and Cornalba (cf. [1]), the scheme $\mathcal{W}_{\mathcal{C}/S}^{1,d}$ is smooth of dimension $= \dim S + \rho(g, 1, d)$ outside $\mathcal{W}_{\mathcal{C}/S}^{2,d}$ and the image of p coincides with the points of S such that $[C_s] \in M_{g,d}^1$. Since the construction can be globalized, the result follows from exhibiting a curve $C \in M_g$ and a base point free $A \in W_d^1(C) \setminus W_d^2(C)$ such that $\dim \ker \mu_{0,A} = -\rho(g, 1, d)$. Segre (cf. [21]) realized such a curve as a particular plane curve.

Now, let us turn our attention to some particular loci $GP_{g,d}^r$ with $\rho(g, r, d) \geq 0$.

If g is even, the locus ${}^bGP_{g,(g+2)/2}^1$ was used by Eisenbud and Harris (cf. [10]) in order to prove that M_g is of general type for $g \geq 24$. It turns out that ${}^b\overline{GP}_{g,(g+2)/2}^1$ is a divisor in M_g which can be alternatively described as the closure of the branch locus of the natural map $H_{g,(g+2)/2} \rightarrow M_g$ from the Hurwitz scheme $H_{g,(g+2)/2}$ parametrizing coverings of \mathbb{P}^1 of degree $(g + 2)/2$ having as source a smooth curve C of genus g . In the next section we will show that ${}^bGP_{g,(g+2)/2}^1$ is dense in $GP_{g,(g+2)/2}^1$.

The component $GP_{g,g-1}^1$ coincides instead with the locus of curves with a vanishing theta-null, that is, having a theta-characteristic whose space of sections has dimension at least 2. By definition, a theta-characteristic on a curve C is a line bundle $A \in \text{Pic}^{g-1}(C)$ which satisfies $A^2 \cong \omega_C$. Any non-singular curve C of genus g has 2^{2g} theta-characteristics corresponding to the points of order 2 in the Jacobian variety $J(C) \simeq \text{Pic}^{g-1}(C)$. Riemann Singularity Theorem (cf. [3, Ch. VI]) implies that the space of sections of a theta characteristic A on C has dimension at least 2 whenever A corresponds to a singular point of the theta divisor $\Theta \simeq W_{g-1}^0(C)$. If this happens, then $[C] \in GP_{g,g-1}^1$; indeed, by the Base Point Free Pencil Trick (cf. [3, p. 126]) the condition $A^2 \cong \omega_C$ prevents the map $\mu_{0,V}$ from being injective for any pencil (A, V) .

Mumford ([20]) was the first to treat theta-characteristics purely algebraically, without referring to the theory of theta-functions. This made it possible to study theta-characteristics on singular curves, which were used by Teixidor in order to prove that $GP_{g,g-1}^1$ is an irreducible divisor in M_g (cf. [17, 23, 24]).

The following result was recently proved by Bruno and Sernesi ([4]) and exhibits some other divisorial components of GP_g :

Theorem 2.2.3. *Let g, r, d be integers such that $0 < 2r \leq d \leq g-1$, $\rho(g, r, d) \geq 0$ and $\rho(g, r+1, d) < 0$. Then any irreducible component of $GP_{g,d}^r$ whose general point does not lie in $M_{g,d}^{r+1}$ has pure codimension 1 inside M_g .*

The condition $\rho(g, r+1, d) < 0$ assures that on a generic curve of genus g every g_d^r is complete. In this situation one can find a modular family $\varphi : \mathcal{C} \rightarrow S$ of smooth curves of genus g not belonging to $M_{g,d}^{r+1}$ such that the induced map $S \rightarrow M_g$ is dominant and finite. The morphism of vector bundles

$$\mu : \mathcal{E}_1 \otimes \mathcal{E}_2 \rightarrow \mathcal{F}$$

globalizing the Petri map is well defined over the relative scheme $p : \mathcal{W}_{\mathcal{C}/S}^{r,d} \rightarrow S$ and the locus

$$GP_{\mathcal{C}/S}^{r,d} := \{s \in S \mid \varphi^{-1}(s) \in GP_{g,d}^r\}$$

coincides with image under p of the degeneracy locus $X_{(r+1)(g-d+r)-1}(\mu)$.

If $X_{(r+1)(g-d+r)-1}(\mu)$ is nonempty, then its codimension inside $\mathcal{W}_{\mathcal{C}/S}^{r,d}$ does not exceed $\rho(g, r, d) + 1$. The finiteness of the fibres of the restriction of p to $X_{(r+1)(g-d+r)-1}(\mu)$ follows from the already mentioned result of Steffen (cf. [22]), which can be applied because p is projective and dominant and the sheaf $(\mathcal{E}_1 \otimes \mathcal{E}_2)^\vee \otimes \mathcal{F}$ is ample relative to p .

Without the condition $\rho(g, r+1, d) < 0$, we could still define the sheaves $\mathcal{E}_1, \mathcal{E}_2$ and \mathcal{F} in the same way but \mathcal{E}_1 and \mathcal{E}_2 would be locally free only when restricted to the open subset $\mathcal{W}_{\mathcal{C}/S}^{r,d} \setminus \mathcal{W}_{\mathcal{C}/S}^{r+1,d}$. Unfortunately, the restriction of p to $\mathcal{W}_{\mathcal{C}/S}^{r,d} \setminus \mathcal{W}_{\mathcal{C}/S}^{r+1,d}$ is not projective and so Steffen's Theorem cannot be applied in this situation.

2.3 Some useful inclusions

In this section we prove some inclusions among different components of GP_g , which enable us to restrict the values of r and d for which the codimension of $GP_{g,d}^r$ must be determined.

Lemma 2.3.1. *For $\rho(g, r-1, d-1) < 0$ and $r > 1$, we have that:*

$$M_{g,d}^r \subset M_{g,d-1}^{r-1} = GP_{g,d-1}^{r-1}.$$

Proof. From any g_d^r we can trivially get a g_{d-1}^{r-1} by subtracting a point outside its base locus. \square

Next result concerns the components $GP_{g,k}^1$:

Lemma 2.3.2. *If g is odd, the following sequence of inclusions holds:*

$$M_{g,2}^1 \subseteq M_{g,3}^1 \subseteq \dots \subseteq M_{g, \frac{g+1}{2}}^1,$$

and $M_{g, \frac{g+1}{2}}^1$ is a Brill-Noether divisor.

Similarly when g is even we have that:

$$M_{g,2}^1 \subseteq M_{g,3}^1 \subseteq \dots \subseteq GP_{g, \frac{g+2}{2}}^1.$$

Proof. Consider $k < \frac{g+1}{2}$ if g is odd and $k < \frac{g+2}{2}$ if g is even. Let $[C] \in M_{g,k}^1$ and A be a complete g_k^1 on C . By defining $A' := A \otimes \mathcal{O}_C(P)$ with P a point outside the base locus of $\omega_C \otimes A^\vee$, one may prove all the inclusions but $M_{g, \frac{g+2}{2}}^1 \subset GP_{g, \frac{g+2}{2}}^1$. When A is a complete $g_{\frac{g}{2}}^1$ on C with base locus B (not necessarily empty), the Base Point Free Pencil Trick implies both

$$\dim \ker \mu_{0,A} = h^0(C, \omega_C \otimes A^{-2} \otimes \mathcal{O}_C(B)) \geq -\rho(g, 1, g/2) = 2$$

and

$$\dim \ker \mu_{0,A'} = h^0(C, \omega_C \otimes A'^{-2} \otimes \mathcal{O}_C(B-P)) \geq 1.$$

Thus A' is a $g_{\frac{g+2}{2}}^1$ on C violating the Gieseker-Petri Theorem and $[C] \in GP_{g, \frac{g+2}{2}}^1$. \square

The following result is a corollary of Theorem 2.2.3. Together with the previous Lemma, it implies that all the loci $GP_{g,k}^1$ such that $\rho(g, 1, k) < 0$ are contained in a divisorial component of GP_g .

Corollary 2.3.3. *In even genus the following equality holds:*

$${}^b\overline{GP}_{g, \frac{g+2}{2}}^1 = GP_{g, \frac{g+2}{2}}^1.$$

Proof. By Lemma 2.3.2, we have that $M_{g, \frac{g}{2}}^1 \subset GP_{g, \frac{g+2}{2}}^1$ and so we can write

$$GP_{g, \frac{g+2}{2}}^1 = {}^b GP_{g, \frac{g+2}{2}}^1 \cup M_{g, \frac{g}{2}}^1,$$

where ${}^b \overline{GP}_{g, \frac{g+2}{2}}^1$ is a divisor on M_g . Furthermore $M_{g, \frac{g}{2}}^1$ is irreducible and of codimension 2 in M_g (cf. [15]). Our goal is to show that $M_{g, \frac{g}{2}}^1 \subset {}^b \overline{GP}_{g, \frac{g+2}{2}}^1$.

Theorem 2.2.3 implies that $GP_{g, \frac{g+2}{2}}^1 \setminus M_{g, \frac{g+2}{2}}^2$ is divisorial, and by Lemma 2.3.1 we know that $M_{g, \frac{g+2}{2}}^2 \subset M_{g, \frac{g}{2}}^1$. It follows that

$$M_{g, \frac{g}{2}}^1 \setminus M_{g, \frac{g+2}{2}}^2 \subset {}^b GP_{g, \frac{g+2}{2}}^1,$$

and the same must be true passing to the closures. If we show that $M_{g, \frac{g}{2}}^1 \setminus M_{g, \frac{g+2}{2}}^2$ is open in $M_{g, \frac{g}{2}}^1$, then the irreducibility of $M_{g, \frac{g}{2}}^1$ implies that $M_{g, \frac{g}{2}}^1 \subset {}^b \overline{GP}_{g, \frac{g+2}{2}}^1$ and we have finished. To end the proof it is enough to remark that the generic curve in $M_{g, \frac{g}{2}}^1$ has a unique $g_{\frac{g}{2}}^1$ (cf. [1]) while a curve inside $M_{g, \frac{g+2}{2}}^2$ has at least a 1-dimensional space of $g_{\frac{g}{2}}^1$'s (all obtained from a $g_{\frac{g+2}{2}}^2$ by the subtraction of a point). \square

Other useful inclusions come from the following fact:

Lemma 2.3.4. *If $\rho(g, r, d) \in \{0, 1\}$, then $M_{g, d+1}^{r+1} \subset GP_{g, d}^r$ and $M_{g, d-1}^r \subset GP_{g, d}^r$.*

Proof. Assume $\rho(g, r, d) = 0$. We fix $[C] \in M_{g, d+1}^{r+1}$ and A a complete g_{d+1}^{r+1} on C . For any $P \in C$ outside the base locus of A , the linear series $A(-P) := A \otimes \mathcal{O}_C(-P)$ is a g_d^r on C and so $G_d^r(C)$ contains

$$C' := \overline{\{A(-P) : P \in C, P \notin \text{bs}(|A|)\}} \cong C.$$

It follows that $\dim T_{A(-P)}(G_d^r(C)) \geq \dim_{A(-P)} G_d^r(C) \geq 1$. By remembering that

$$\dim T_{A(-P)}(G_d^r(C)) = \rho(g, r, d) + \dim \ker \mu_{0, A(-P)} = \dim \ker \mu_{0, A(-P)},$$

one deduces that $A(-P)$ does not satisfy the Gieseker-Petri Theorem. Analogously, given $[C] \in M_{g, d-1}^r$ and A a complete g_{d-1}^r on C , the variety $G_d^r(C)$ contains the locus

$$C'' := \overline{\{A(P) : P \in C, P \notin \text{bs}(|\omega_C \otimes A^\vee|)\}} \cong C$$

and, reasoning as above, one proves that $[C] \in GP_{g, d}^r$.

For $\rho(g, r, d) = 1$, we consider $[C] \in M_{g, d-1}^r$ and A a complete g_{d-1}^r on C . The definition of C'' is the same. Since we can assume that $\dim G_d^r(C) = 1$ (otherwise we could soon conclude that $[C] \in GP_{g, d}^r$), it follows that C'' is an irreducible component of

2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$

$G_d^r(C)$. As C must have a base point free g_d^r , there exist components of $G_d^r(C)$ different from C'' . By the Connectedness Theorem (cf.[3, p. 212]), $G_d^r(C)$ is connected. It follows that $G_d^r(C)$ is singular and so $[C] \in GP_{g,d}^r$. We proceed very similarly if $[C] \in M_{g,d+1}^{r+1}$. \square

2.4 Proof of Theorem 2.1.1 in genus 9, 10, 11

In this section we prove that, for genus $g \in \{9, 10, 11\}$, the Gieseker-Petri locus GP_g is of pure codimension 1 inside M_g .

Let us fix $g = 9$. For $r \in \{4, 3\}$ and $2r \leq d \leq 8$ and for $r = 2$ and $4 \leq d \leq 6$, the Brill-Noether number $\rho(g, r-1, d-1)$ is negative and so, by Lemma 2.3.1, we can restrict our analysis to the components $GP_{9,d}^2$ and $GP_{9,k}^1$ for $d \in \{7, 8\}$ and $2 \leq k \leq 8$. Moreover, Lemma 2.3.2 implies that $M_{9,k}^1$ is contained in the Brill-Noether divisor $M_{9,5}^1$ for $k \leq 4$.

Since $\rho(9, 2, 7) < 0$, we now study $M_{9,7}^2$. Given $[C] \in M_{9,7}^2$, if we assume that C does not lie in $M_{9,5}^1$, then any g_7^2 on C is base point free and defines an embedding

$$\phi : C \rightarrow \Gamma \subset \mathbb{P}^2,$$

where Γ is a plane curve of degree 7 and genus 9. By the Genus Formula it follows that Γ is singular, which is a contradiction.

Regarding the component $GP_{9,8}^2$, we note that $\rho(9, 2, 8) = 0$ and $\rho(9, 3, 8) < 0$, so Theorem 2.2.3 implies that $GP_{9,8}^2 \setminus (M_{9,8}^3 \cap GP_{9,8}^2)$ is divisorial. We do not need to study $M_{9,8}^3 \cap GP_{9,8}^2$ separately because, by Lemma 2.3.1, the inclusion $M_{9,8}^3 \subseteq M_{9,7}^2$ holds.

Let us consider the components $GP_{9,k}^1$ for $k \in \{6, 7, 8\}$. For $k \in \{6, 7\}$ we have that $\rho(9, 1, k) > 0$ and $\rho(9, 2, k) < 0$ and so the locus $GP_{9,k}^1 \setminus (GP_{9,k}^1 \cap M_{9,k}^2)$ is divisorial. As $GP_{9,8}^1$ is the irreducible divisor consisting of curves with a vanishing theta-null, Theorem 2.1.1 is proved in genus 9.

Before dealing with the case of genus 10, we prefer to treat the case of genus 11, which is very similar to the one we have just studied. As before, by applying Lemma 2.3.1 and Lemma 2.3.2 we reduce to considering the components $GP_{11,d}^2$ and $GP_{11,k}^1$ for $8 \leq d \leq 10$ and $7 \leq k \leq 10$.

We can prove that $M_{11,8}^2$ is contained in the Brill-Noether divisor $M_{11,6}^1$ simply by remarking that any g_8^2 on a genus 11 curve $[C] \notin M_{11,6}^1$ is base point free and defines an embedding

$$\phi : C \rightarrow \Gamma \subset \mathbb{P}^2.$$

We get a contradiction because Γ is a plane curve of degree 8 and genus 11 and so it must be singular by the Genus Formula.

As for the other components, the locus $M_{11,9}^2$ is a Brill-Noether divisor, while $GP_{11,10}^2$ is divisorial outside its intersection with $M_{11,10}^3$ as $\rho(11, 2, 10) > 0$ and $\rho(11, 3, 10) < 0$. Theorem 2.2.3 can be applied in order to prove that $GP_{11,k}^1 \setminus (M_{11,k}^2 \cap GP_{11,k}^1)$ is divisorial

for $7 \leq k \leq 9$, too. The component $GP_{11,10}^1$ is the irreducible divisor of curves with a vanishing theta-null and so Theorem 2.1.1 is proved in genus 11.

We now deal with the case of genus 10. As above, by Lemmas 2.3.1 and 2.3.2, the only components of GP_{10} we have to consider are $GP_{10,d}^2$ and $GP_{10,k}^1$ for $7 \leq d \leq 9$ and $7 \leq k \leq 9$.

As $\rho(10, 1, 6) = 0$, Lemma 2.3.4 implies that $M_{10,7}^2 \subset GP_{10,6}^1$. Moreover, $\rho(10, 2, 9) = 1$ and so Lemma 2.3.4 implies that $M_{10,8}^2 \subset GP_{10,9}^2$, too. Since $\rho(10, 3, 9) < 0$, the locus $GP_{10,9}^2$ is divisorial outside $M_{10,9}^3$. In this case we have to study the component $M_{10,9}^3$ separately as our remarks imply only that $M_{10,9}^3 \subseteq M_{10,8}^2 \subseteq GP_{10,9}^2$. We postpone the study of $M_{10,9}^3$. For $k \in \{7, 8\}$, the locus $GP_{10,k}^1 \setminus (GP_{10,k}^1 \cap M_{10,k}^2)$ is divisorial because $\rho(10, 2, k) < 0$, while $GP_{10,9}^1$ is the irreducible divisor consisting of curves with a vanishing theta-null.

In order to end the proof of Theorem 2.1.1 in genus 10, we now study $M_{10,9}^3$. We consider $[C] \in M_{10,9}^3$ and L a g_3^1 on C . We can assume $[C] \notin M_{10,8}^3$ and so L , being base point free, defines a morphism $\phi : C \rightarrow \Gamma \subset \mathbb{P}^3$. Furthermore, we can assume that $[C] \notin M_{10,7}^2$, which forces ϕ to be an embedding. Therefore C can be seen as a curve of genus 10 and degree 9 in \mathbb{P}^3 . By the classification of curves in \mathbb{P}^3 , we know that C is either a curve of type $(3, 6)$ on a non singular quadric surface S or the intersection of two cubic surfaces (cf. [18] Example 6.4.3. chp.IV). In the first case the lines of type $(0, 1)$ on S cut out a g_3^1 on Γ . The second case is treated in the following lemma:

Lemma 2.4.1. *Let $[C] \in \mathcal{M}_{10}$ be the intersection of two cubic surfaces X, Y in \mathbb{P}^3 . Then $[C] \in GP_{10,6}^1$.*

Proof. It is classically known that X is isomorphic to the blow-up of \mathbb{P}^2 in 6 points P_1, \dots, P_6 . We denote by $\pi : X \rightarrow \mathbb{P}^2$ the projection and by E_i the exceptional divisors. The Picard group $\text{Pic}(X) \cong \mathbb{Z}^7$ is generated by l, e_1, e_2, \dots, e_6 , where l is the class of the strict transform of a line in \mathbb{P}^2 and e_i is the class of E_i . The class of the hyperplane section is $h = 3l - \sum e_i$, while the class of the canonical divisor is

$$K_X \sim -h = -3l + \sum e_i.$$

As C lies on another cubic surface Y , then

$$C \sim 3h = 9l - 3 \sum e_i,$$

namely C is the strict transform of a plane curve \tilde{C} of degree 9 with 6 triple points P_1, \dots, P_6 . The pencil of cubics through P_1, \dots, P_6 with a double point in P_1 cuts out a g_6^1 on \tilde{C} . The strict transforms of these cubics cut out on C the linear series

$$A := \mathcal{O}_C(3l - \sum_{i \neq 1} e_i - 2e_1).$$

In order to check that A is a g_6^1 on C , we tensor with $\mathcal{O}_X(3l - \sum_{i \neq 1} e_i - 2e_1)$ the exact

2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$

sequence

$$0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0,$$

getting

$$0 \rightarrow \mathcal{O}_X(-6l + 2 \sum_{i \neq 1} e_i + e_1) \rightarrow \mathcal{O}_X(3l - \sum_{i \neq 1} e_i - 2e_1) \rightarrow \mathcal{O}_C(3l - \sum_{i \neq 1} e_i - 2e_1) \rightarrow 0.$$

As $6l - 2 \sum_{i \neq 1} e_i - e_1$ is ample (cf. [18] Cor.4.13 chap.V), Kodaira Vanishing Theorem implies that $h^i(X, \mathcal{O}_X(-6l + 2 \sum_{i \neq 1} e_i + e_1)) = 0$ for $i = 0, 1$. It follows that

$$\begin{aligned} h^0(C, \mathcal{O}_C(3l - \sum_{i \neq 1} e_i - 2e_1)) &= h^0(X, \mathcal{O}_X(3l - \sum_{i \neq 1} e_i - 2e_1)) &= \\ &= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{\mathbb{P}^2}(-\sum_{i \neq 1} P_i - 2P_1)) &= \\ &= 2 \end{aligned}$$

and this is enough to conclude that A is a pencil on C ; it is trivial to check that its degree is 6.

By the Base Point Free Pencil Trick we have that

$$\ker \mu_{0,A} \cong H^0(C, \omega_C \otimes A^{-2}) = H^0(C, \mathcal{O}_C(2e_1)).$$

As $\mathcal{O}_C(2e_1)$ is effective, it follows that $[C] \in GP_{10,6}^1$. □

Remark 1. The previous Lemma can also be proved by using the results of [19]. Curves of genus 10 which are the complete intersection of two cubic surfaces in \mathbb{P}^3 are the only curves of Clifford dimension 3. Martens proved that such curves are 6-gonal and carry a one-dimensional family of g_6^1 . Since $\rho(10, 1, 6) = 0$, this is enough to conclude that they lie in $GP_{10,6}^1$.

It is natural to ask whether all curves of Clifford dimension greater than 1 lie in a divisorial component of the Gieseker-Petri locus. Curves of Clifford dimension 2 are smooth plane curves of degree $d \geq 5$. Their gonality is $d - 1$ and there is a one-dimensional family of pencils computing it. As $\rho\left(\binom{d-1}{2}, 1, d-1\right) \leq 0$ for $d \geq 5$, Lemma 2.3.2 implies that they lie in the Brill-Noether divisor $M_{g, \frac{g+1}{2}}^1$ when $g = \binom{d-1}{2}$ is odd, and in the irreducible divisor $GP_{g, \frac{g+2}{2}}^1$ when g is even.

It was conjectured in [12] that, if a curve C has Clifford dimension $r > 3$, then one has $g(C) = 4r - 2$, $\text{gon}(C) = 2r$ and there is a one-dimensional family of pencils computing the gonality (this conjecture was proved in [12] for $r \leq 9$). Since $\rho(4r - 2, 1, 2r) = 0$, such curves lie in the divisor $GP_{g, \frac{g+2}{2}}^1 = GP_{4r-2, 2r}^1$.

2.5 Proof of Theorem 2.1.1 in genus 12, 13

The situation in genus 12 and 13 is slightly more complicated as there is a component in GP_g which cannot be studied using the methods explained in the previous sections.

2.5 Proof of Theorem 2.1.1 in genus 12, 13

In genus 12, by Remarks 2.3.1 and 2.3.2, we have to analyze only the components $M_{12,11}^3$, $GP_{12,d}^2$ for $8 \leq d \leq 11$ and $GP_{12,k}^1$ for $8 \leq k \leq 11$. Since $\rho(12, 2, 10) = 0$, Lemma 2.3.4 implies that both $M_{12,11}^3$ and $M_{12,9}^2$ are contained in $GP_{12,10}^2$. Lemma 2.3.4 can also be used in order to show that $M_{12,8}^2 \subset GP_{12,7}^1$; indeed, $\rho(12, 1, 7) = 0$.

As $\rho(12, 3, d) < 0$ for $d \in \{10, 11\}$, the loci $GP_{12,10}^2$ and $GP_{12,11}^2$ are divisorial outside their intersection with $M_{12,10}^3$ and $M_{12,11}^3$ respectively. We have to study $M_{12,10}^3$ separately because our remarks only imply that $M_{12,10}^3 \subset M_{12,9}^2 \subset GP_{12,10}^2$.

Given $[C] \in M_{12,10}^3$, we can suppose that $[C] \notin M_{12,8}^2$ and so any g_{10}^3 on C is base point free and defines an embedding $\phi : C \rightarrow \Gamma \subset \mathbb{P}^3$. It can be seen that Γ has ten 4-secant lines (cf. [3], p. 351), each of which corresponds to a g_6^1 on it.

Theorem 2.2.3 can be applied in order to show that the locus $GP_{12,k}^1$ is divisorial outside $M_{12,k}^2$ for $k \in \{8, 9\}$. The component $GP_{12,11}^1$ is an irreducible divisor. We postpone the study of $GP_{12,10}^1$ to the end of the section.

The situation in genus 13 is very similar to that in genus 12. By Remarks 2.3.1 and 2.3.2, we reduce to considering $M_{13,12}^3$, $GP_{13,d}^2$ for $9 \leq d \leq 12$ and $GP_{13,k}^1$ for $8 \leq k \leq 12$.

As $\rho(13, 2, 11) = 1$, Lemma 2.3.4 implies that both $M_{13,12}^3$ and $M_{13,10}^2$ are contained in $GP_{13,11}^2$.

Concerning $M_{13,9}^2$, any g_9^2 on a genus 13 curve $[C] \notin M_{13,7}^1$ defines an embedding $\phi : C \rightarrow \Gamma \subset \mathbb{P}^2$. We get a contradiction because the Genus Formula forces Γ to be singular.

The components $GP_{13,11}^2$ and $GP_{13,12}^2$ are divisorial outside $M_{13,11}^3$ and $M_{13,12}^3$ respectively. As before we have to study $M_{13,11}^3$ separately. Given $[C] \in M_{13,11}^3$ such that $[C] \notin M_{13,9}^2$, by taking the 4-secant lines to the space model of C corresponding to any $l \in G_{11}^3(C)$, one shows that C has a g_7^1 .

Regarding the other components, the locus $GP_{13,k}^1$ is divisorial outside its intersection with $M_{13,k}^2$ for $k \in \{8, 9, 10\}$, while $GP_{13,12}^1$ is an irreducible divisor. Therefore Theorem 2.1.1 is proved also in genus 13 if we are able to verify that the component $GP_{g,g-2}^1$ is divisorial. In order to show this, we generalize a result of Castorena (cf. [6]) as follows.

We consider curves $[C] \in GP_{g,g-2}^1$ such that for any $A \in G_{g-2}^1(C)$ for which $\mu_{0,A}$ is not injective the following are satisfied:

1. A is primitive.
2. The morphism $\phi := \phi_{\omega_C \otimes A^\vee}$ is birational.

We remark that the first condition is satisfied if

$$[C] \notin GP_{g,g-3}^1 \cup GP_{g,g-2}^2 \cup GP_{g,g-1}^2,$$

because if A were not complete (respectively not base point free), this would imply $[C] \in GP_{g,g-2}^2$ (resp. $[C] \in GP_{g,g-3}^1$). Similarly, if $\omega_C \otimes A^\vee$ is not base point free,

2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$

then $[C] \in GP_{g,g-1}^2$. We prove the following result:

Proposition 2.5.1. *Let $Z_g \subset GP_{g,g-2}^1$ be the locus consisting of curves C inside $GP_{g,g-2}^1$ such that, if $A \in G_{g-2}^1(C)$ satisfies $\ker \mu_{0,A} \neq 0$, then A is primitive and the morphism $\phi := \phi_{\omega_C \otimes A^\vee}$ is birational. The scheme Z_g has pure codimension 1 in M_g outside its intersection with the hyperelliptic locus and the divisor $GP_{g,g-1}^1$.*

It is clear that Z_g is open in $GP_{g,g-2}^1$ but in general it is not dense. Indeed, given an irreducible component $W \subset GP_{g,g-2}^1$, it may happen that the general element of W lies in $GP_{g,g-3}^1 \cup GP_{g,g-2}^2 \cup GP_{g,g-1}^2$ and that it does not satisfy condition 1. Analogously, we could have that, if $[C] \in W$ is general, there exists a primitive $A \in G_{g-2}^1(C)$ such that $\ker \mu_{0,A} \neq 0$ and $\phi_{\omega_C \otimes A^\vee}$ defines a finite covering of a plane curve of degree strictly less than g . In order to prove Proposition 2.5.1 we need the following:

Lemma 2.5.2. *If $[C] \in Z_g$ and A is a g_{g-2}^1 on C such that $\ker \mu_{0,A} \neq 0$, then A is the pullback to C of the pencil cut out on $\Gamma := \phi_{\omega_C \otimes A^\vee}(C)$ by the lines through a singular point x . In particular, x is a double point of Γ and*

$$\omega_C \otimes A^{-2} = \frac{1}{k} \phi^* \mathcal{O}_\Gamma(x),$$

where k is the number of blow-ups necessary to desingularize Γ in x (e.g., if x is a tacnode, then $k = 2$).

Proof. The statement follows directly from the Base Point Free Pencil Trick, which implies that $\omega_C \otimes A^{-2} = \phi^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes A^\vee$ has at least a 1-dimensional space of sections. The point x must be a double point because A is base point free. \square

We can now prove the following fact:

Lemma 2.5.3. *If $[C] \in Z_g$, $[C] \notin GP_{g,g-1}^1$ and C is not hyperelliptic, then there exists only a finite number of $A \in W_{g-2}^1(C)$ such that $\mu_{0,A}$ is not injective.*

Proof. We recall and adapt the proof of Castorena, referring to [6] for further details. Given A a g_{g-2}^1 on C with $\ker \mu_{0,A} \neq 0$, we have that

$$\omega_C \otimes A^{-2} = \frac{1}{k} \phi^* \mathcal{O}_\Gamma(x) = \mathcal{O}_C(P + Q),$$

and we can assume $P \neq Q$ because otherwise $A \otimes \mathcal{O}_C(P)$ would be a theta characteristic with a 2-dimensional space of sections, thus contradicting the hypothesis $[C] \notin GP_{g,g-1}^1$. We remark that asking that $P \neq Q$ is equivalent to requiring that x be not a cusp of any order. As C is not hyperelliptic, we obtain that $h^0(C, \omega_C \otimes \mathcal{O}_C(-P - Q)) = g - 2$ and $h^0(C, \mathcal{O}_C(P + Q)) = 1$. It follows that A^2 lies in the intersection of the following two subvarieties of $\text{Pic}^{2g-4}(C)$:

$$\begin{aligned} X_1 &:= \{A^2 \mid A \in W_{g-2}^1(C)\}, \\ X_2 &:= \{\omega_C \otimes \mathcal{O}_C(-P - Q) \mid P, Q \in C\} \subset W_{2g-4}^{g-3}(C). \end{aligned}$$

We have to show that A^2 is an isolated point in $X_1 \cap X_2$, which amounts to proving that $T_{A^2}(X_1) \cap T_{A^2}(X_2) = \{0\}$ in $H^1(C, \mathcal{O}_C) = T_{A^2}(\text{Pic}^{2g-2r}(C))$. Such condition is equivalent to requiring that $T_{A^2}(X_1)^\perp + T_{A^2}(X_2)^\perp$ generates the whole space $H^0(C, \omega_C) = T_{A^2}(\text{Pic}^{2g-4}(C))^\perp$. In fact, the following holds:

$$\dim T_{A^2}(X_1)^\perp = \dim \text{Im } \mu_{0,A} = 5.$$

Since μ_{0,A^2} is injective, $T_{A^2}(X_2)^\perp \simeq \text{Im } \mu_{0,A^2} \simeq H^0(C, \omega_C \otimes \mathcal{O}_C(-P - Q))$, which is $(g-2)$ -dimensional.

We claim that $\text{Im } \mu_{0,A} \cap H^0(C, \omega_C \otimes \mathcal{O}_C(-P - Q))$ coincides with the image of the restriction of $\mu_{0,A}$ to $H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee \otimes \mathcal{O}_C(-P - Q))$. This enables us to conclude that $\dim T_{A^2}(X_1)^\perp \cap T_{A^2}(X_2)^\perp = 3$, since the space

$$H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee \otimes \mathcal{O}_C(-P - Q)) \simeq H^0(C, A) \otimes H^0(C, A)$$

is 4-dimensional and contains the 1-dimensional kernel of $\mu_{0,A}$. Our claim follows by the fact that x becomes a node after $k-1$ blow-ups. Indeed, if $\phi_{k-1} : X_{k-1} \rightarrow \mathbb{P}^2$ denotes the composition of these blow-ups and C_{k-1} is the strict transform of Γ under ϕ_{k-1} , then there exist two distinct lines l_1 and l_2 in \mathbb{P}^2 whose strict transforms in X_{k-1} are the two tangent lines to C_{k-1} in $(\phi_{k-1}|_{C_{k-1}})^*(x)$. The linear system $|L \otimes \mathcal{O}_C(-P)|$ (resp. $|L \otimes \mathcal{O}_C(-Q)|$) contains a unique divisor D_1 (resp. D_2). The divisors D_1 and D_2 are distinct since one of them is cut out by the strict transform under ϕ of l_1 and the other by the strict transform of l_2 ; moreover, D_1 does not contain Q and D_2 does not contain P . This implies that $H^0(C, A \otimes \mathcal{O}_C(-P - Q)) = 0$ and, since

$$\begin{aligned} H^0(C, \omega_C \otimes A^\vee \otimes \mathcal{O}_C(-P)) &= H^0(C, \omega_C \otimes A^\vee \otimes \mathcal{O}_C(-P - Q)) \\ &= H^0(C, \omega_C \otimes A^\vee \otimes \mathcal{O}_C(-Q)), \end{aligned}$$

our claim follows. \square

Proof of Proposition 2.5.1. Let $[C] \in Z_g$ be a non hyperelliptic curve with no vanishing theta-null. One may find a neighborhood $U \subset \mathcal{M}_g$ of C , intersecting neither the hyperelliptic locus nor the divisor $GP_{g,g-1}^1$, such that there exists a finite ramified covering $\pi : \tilde{U} \rightarrow U$, a universal curve $\varphi : \Gamma_{\tilde{U}} \rightarrow \tilde{U}$ and a variety $\mathcal{G}_{g-2}^1 \xrightarrow{\xi} \tilde{U}$ proper over \tilde{U} which parametrizes pairs $(C, (A, V))$ with $[C] \in \tilde{U}$ and (A, V) a g_{g-2}^1 on $\varphi^{-1}(C)$. Up to restricting U , we can also assume that $U \cap GP_{g,g-2}^1 \subset Z_g$. The scheme \mathcal{G}_{g-2}^1 is smooth of dimension $\rho(g, 1, g-2) + \dim \mathcal{M}_g$ (cf. [2]). We define the following subvariety of \mathcal{G}_{g-2}^1 :

$$\tilde{Z}_g := \{(C, A) \in \mathcal{G}_{g-2}^1 \mid [C] \in \pi^{-1}(Z_g \cap U), \ker \mu_{0,A} \neq 0\}.$$

Lemma 2.5.3 implies that the fibre of the projection from \tilde{Z}_g on $Z_g \cap U$ is finite. For any $(C, L) \in \tilde{Z}_g$, the curve C is not hyperelliptic and so $\dim \text{Im } \mu_{0,A} = 5$. Locally the Petri map defines a homomorphism μ of vector bundles on \mathcal{G}_{g-2}^1 and \tilde{Z}_g can be identified

2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$

with the fifth degeneracy locus of μ . By the fact that each irreducible component of \tilde{Z}_g has codimension $\leq \rho(g, 1, g-2) + 1$ in \mathcal{G}_{g-2}^1 and by the finiteness of the fibres of $\pi \circ \xi$ over the points of $\pi \circ \xi(\tilde{Z}_g)$, we can deduce that each component of $Z_g \cap U$ has codimension at most 1 in U . It must be 1 because of the Gieseker-Petri Theorem. \square

Remark 2. Lemma 2.5.2 can be generalized as follows. Fix $2 < r < (g+6)/4$ such that $\rho(g, 1, g-r) > 0$. If $C \in GP_{g, g-r}^1$ has maximal gonality and A is a primitive g_{g-r}^1 on C such that $\ker \mu_{0,A} \neq 0$ and $\phi_{\omega_C \otimes A^\vee}$ is birational, then A is the pullback to C of the pencil cut out on $\Gamma := \phi_{\omega_C \otimes A^\vee}(C)$ by the hyperplanes containing an $(r-2)$ -plane $\pi \subset \mathbb{P}^r$, which is $(2r-2)$ -secant to Γ . In order to gain a statement analogous to that of Lemma 2.5.3, we need some assumptions on Γ . If π cuts out a divisor D_{2r-2} on Γ consisting of $2r-2$ distinct smooth points, then the hypothesis on the gonality of C assures that $\ker \mu_{0,A}$ is 1-dimensional and μ_{0,A^2} is injective. It follows that A^2 lies in two subvarieties X_1 and X_2 of $\text{Pic}^{2g-2r}(C)$ whose definition is analogous to the one given above. However, in order to show that A^2 is an isolated point in $X_1 \cap X_2$, we need to assume that there does not exist a k -plane $\pi_1 \subset \mathbb{P}^r$ for some $k \leq r-3$ and a hyperplane $H \supset \pi$ such that H is tangent to Γ in all the points of D_{2r-2} not lying in π_1 .

Proposition 2.5.1 implies the following:

Corollary 2.5.4. *The locus GP_{13} has pure codimension 1 in M_{13} .*

Proof. By the above discussion we should only study the component $GP_{13,11}^1$. Given $[C] \in GP_{13,11}^1$, assume that $[C]$ does not lie in $GP_{13,10}^1 \cup GP_{13,11}^2 \cup GP_{13,12}^2$. In particular, condition 1 is satisfied for any $A \in G_{13}^1(C)$ for which the Gieseker-Petri Theorem fails. Moreover, $\omega_C \otimes A^\vee$ cannot be composed with any involution and so condition 2 is satisfied, too. It follows that $[C] \in Z_g$ and so Proposition 2.5.1 is enough to conclude. \square

Next we turn to the case of genus 12. Given $[C] \in GP_{12,10}^1$ such that condition 1 is satisfied for any $A \in G_{10}^1(C)$ with $\ker \mu_{0,A} \neq 0$, it could still happen that some of the above $A \in W_{10}^1(C)$ violate condition 2, that is, $\phi_{\omega_C \otimes A^\vee}$ is not birational. We prove the following:

Theorem 2.5.5. *Let $[C] \in GP_{12,10}^1$ and let us assume that condition 1 is satisfied for any A in $G_{10}^1(C)$ such that $\ker \mu_{0,A} \neq 0$. If for one of such $A \in W_{10}^1(C)$ the morphism $\omega_C \otimes A^\vee$ defines a finite covering of a plane curve Γ of degree strictly less than 12, then $[C]$ lies in $GP_{12,7}^1 \cup GP_{12,8}^1$.*

Proof. Let $[C] \in GP_{12,10}^1$ be as in the hypothesis and A be a g_{10}^1 on C for which the Gieseker-Petri Theorem fails. If $\phi := \phi_{\omega_C \otimes A^\vee} : C \rightarrow \Gamma \subset \mathbb{P}^2$ is not birational, then it is a finite covering of degree 6, 4, 3 or 2. We analyze these cases.

(I): $\deg \phi_{\omega_C \otimes A^\vee} = 6$. In this case Γ is rational and so C has a g_6^1 .

(II): $\deg \phi_{\omega_C \otimes A^\vee} = 3$. Then Γ has degree 4 and genus at most 3. If $g(\Gamma) < 3$, then Γ has at least one singular point and by taking the lines through it one sees that Γ is hyperelliptic and so C has a g_6^1 .

Let us consider $g(\Gamma) = 3$. As the triple cover is induced by $\omega_C \otimes A^\vee$, it follows that $\omega_C \otimes A^\vee = \phi^* \mathcal{O}_\Gamma(1) = \phi^* \omega_\Gamma$ and so $A = \mathcal{O}_C(R)$, where R is the ramification locus. The Base Point Free Pencil Trick thus implies that

$$\ker \mu_{0,A} \simeq H^0(C, \omega_C \otimes \mathcal{O}_C(-2R)) \simeq H^0(C, \phi^* \mathcal{O}_\Gamma(1) \otimes \mathcal{O}_C(-R)).$$

If this were not zero, then there would exist a divisor D on Γ , $\mathcal{O}_\Gamma(D) = \mathcal{O}_\Gamma(1)$, such that $\phi^* D - R \geq 0$. This would imply that D contains the branch locus B but this is impossible because $\deg B \geq \frac{1}{2} \deg R = 5$ while $\deg D = 4$.

(III): $\deg \phi_{\omega_C \otimes A^\vee} = 4$. The curve Γ has degree 3 and so it is either a rational curve or a smooth elliptic curve. In the first case C has a g_4^1 and lies in $GP_{12,7}^1$.

If Γ is elliptic, then we have that $\omega_C \otimes A^\vee = \phi^* \mathcal{O}_\Gamma(1)$ and

$$A = \phi^*(\omega_\Gamma \otimes \mathcal{O}_\Gamma(-1)) \otimes \mathcal{O}_C(R) = \phi^* \mathcal{O}_\Gamma(-1) \otimes \mathcal{O}_C(R).$$

It follows that

$$\ker \mu_{0,A} \simeq H^0(C, \mathcal{O}_C(R) \otimes (\mathcal{O}_C(R) \otimes \phi^* \mathcal{O}_\Gamma(-1))^{-2}) = H^0(\phi^* \mathcal{O}_\Gamma(2) \otimes \mathcal{O}_C(-R)).$$

This is nonzero whenever there exists a divisor D on Γ such that $\mathcal{O}(D) = \mathcal{O}_\Gamma(2)$ and $\phi^* D - R \geq 0$. This never happens because D has degree 6 and it should contain the base locus B , whose degree is at least $\frac{1}{3} \deg R > 7$.

(IV): $\deg \phi_{\omega_C \otimes A^\vee} = 2$. The degree of Γ is 6 and by the Riemann-Hurwitz Formula it follows that $g(\Gamma) \leq 6$. We can assume that Γ has only double points as singularities because otherwise Γ has a g_k^1 for some $k \leq 3$ and Lemma 2.3.2 implies that $[C] \in GP_{12,7}^1$. If $g(\Gamma) \leq 4$, it is easy to check that Γ has always a g_3^1 and $[C] \in GP_{12,7}^1$. As a consequence the only two cases that require a more detailed analysis are $g(\Gamma) = 5$ and $g(\Gamma) = 6$.

Let us consider the case where Γ is a plane sextic of genus 5. We can assume that the singularities of Γ are 5 double points P_1, \dots, P_5 . Some of the P_i 's may coincide; indeed, if we need k blow-ups in order to desingularize Γ in P_i , then this point appears k times in the list. We denote by x_i, y_i the counterimage of P_i under the normalization map $p : Y \rightarrow \Gamma$. Denoting by B and R the branch locus and the ramification locus respectively, the Riemann-Hurwitz Formula implies that both B and R have degree 6. The double covering $f : C \rightarrow Y$ induced by ϕ is given by means of a divisor η on Y of degree -3 , which satisfies $2\eta = -B$ and $f_* \mathcal{O}_C = \mathcal{O}_Y \oplus \mathcal{O}_Y(\eta)$. As $\text{Pic}^{-3}(Y) = Y - Y_4$, we can write $\eta = x - D_4$.

We consider the divisor $f^*(D_4) \in \text{Pic}^8(C)$. We can assume that

$$h^0(C, \mathcal{O}_C(f^* D_4)) = h^0(Y, \mathcal{O}_Y(D_4)) + h^0(Y, \mathcal{O}_Y(D_4 + \eta)) = 2,$$

because otherwise we can conclude that $[C] \in \mathcal{M}_{12,8}^2 \subset GP_{12,7}^1$. We would like to prove that $\ker \mu_{0, \mathcal{O}_C(f^* D_4)} \neq 0$, which implies $[C] \in GP_{12,8}^1$. By the Base Point Free Pencil Trick

2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$

we know that $\ker \mu_{0, \mathcal{O}_C(f^*D_4)} \cong H^0(C, \omega_C \otimes \mathcal{O}_C(f^*D_4)^{-2})$, and its dimension equals

$$h^0(C, f^*(\omega_Y \otimes \mathcal{O}_Y(-\eta - 2D_4))) = h^0(Y, \omega_Y \otimes \mathcal{O}_Y(-\eta - 2D_4)) + h^0(Y, \omega_Y \otimes \mathcal{O}_Y(-2D_4));$$

here we have used that $\omega_C = f^*(\omega_Y \otimes \mathcal{O}_Y(-\eta))$.

Since $h^0(Y, \omega_Y \otimes \mathcal{O}_Y(-2D_4)) \neq 0$ whenever D_4 is a theta characteristic on Y , our goal is to show that $h^0(Y, \omega_Y \otimes \mathcal{O}_Y(-\eta - 2D_4)) > 0$. As

$$\omega_Y \otimes \mathcal{O}_Y(-\eta - 2D_4) = \mathcal{O}_Y(3)(-x_1 - y_1 - \dots - x_5 - y_5 - D_4 - x),$$

we need to prove the existence of a plane cubic passing through the points $P_1, P_2, P_3, P_4, P_5, p(x), p(z_1), p(z_2), p(z_3), p(z_4)$, where $D_4 = z_1 + \dots + z_4$.

We can assume that every g_6^2 on Y is base point free and not composed with an involution and that every plane model of Y as a sextic has only double points as singularities (otherwise Y would have a g_3^1 and C a g_6^1); the same is true for all the curves in a neighborhood U of Y in M_5 . Up to shrinking U , we can assume the existence of a proper morphism $\xi : \mathcal{G}_6^2 \rightarrow U$, where \mathcal{G}_6^2 parametrizes couples (Y', l') , with $[Y'] \in U$ and l' a g_6^2 on Y' . We denote by $V_{5,6}$ the variety of irreducible plane curves of degree 6 and genus 5 and by $m : V_{5,6} \rightarrow M_5$ the natural morphism. The locus $m^{-1}(U)$ can be seen as a $PGL(2)$ -bundle on \mathcal{G}_6^2 parametrizing couples $((Y', l'), \mathcal{B}')$ with $(Y', l') \in \mathcal{G}_6^2$ and \mathcal{B}' a frame of l' . Indeed, giving l' and \mathcal{B}' is equivalent to fixing a plane model of Y' . We denote by $p_1 : m^{-1}(U) \rightarrow \mathcal{G}_6^2$ the natural morphism. The restriction $m_U : m^{-1}(U) \rightarrow U$ coincides with the composition $\xi \circ p_1$ and it is proper because both ξ and p_1 are. Denoting by $\pi : M_{5,5} \rightarrow M_5$ the forgetful map, the morphism

$$m_1 : m^{-1}(U) \times_U \pi^{-1}(U) \rightarrow \pi^{-1}(U)$$

is proper because of the invariance of properness under base extension. A point of $m^{-1}(U) \times_U \pi^{-1}(U)$ is of the form $(\Gamma', (Y', z'_1, \dots, z'_5))$, where Y' is the normalization of Γ' .

We remark that $m^{-1}(U) \times_U \pi^{-1}(U)$ has dimension equal to

$$\dim \pi^{-1}(U) + \rho(5, 2, 6) + \dim PGL(2) = \dim \pi^{-1}(U) + 10.$$

Let

$$\mathcal{E} := H^0(\mathcal{O}_{\mathbb{P}^2}(3)) \times (m^{-1}(U) \times_U \pi^{-1}(U))$$

be the trivial bundle on $m^{-1}(U) \times_U \pi^{-1}(U)$ and let us define \mathcal{F} to be the bundle on $m^{-1}(U) \times_U \pi^{-1}(U)$ with fibre over $(\Gamma', (Y', z'_1, \dots, z'_5))$ being the space

$$H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{\Delta_{\Gamma'}}) \oplus \bigoplus_{i=1}^5 H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{\phi'(z'_i)}),$$

where $\Delta_{\Gamma'}$ is the scheme of all singular points of Γ' and $\phi' : Y' \rightarrow \Gamma'$ denotes the nor-

malization map. If $\Gamma' \in m^{-1}(U)$ is generic, this space is

$$H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{P'_1}) \oplus \dots \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{P'_5}) \oplus \bigoplus_{i=1}^5 H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{\phi'(z'_i)}),$$

where P'_1, \dots, P'_5 are the nodes of Γ' . We consider the evaluation map $F : \mathcal{E} \rightarrow \mathcal{F}$. Both \mathcal{E} and \mathcal{F} have rank 10 and so the degeneracy locus $X(F)$, if nonempty, has codimension 1 in $m^{-1}(U) \times_U \pi^{-1}(U)$.

In order to show that $X(F) \neq \emptyset$ we observe that, given a cubic $\Gamma_3 \subset \mathbb{P}^2$ and P_1, \dots, P_{10} ten points on it, one can always find a sextic $\Gamma_6 \subset \mathbb{P}^2$ passing through P_6, \dots, P_{10} and having nodes in P_1, \dots, P_5 (because there exists a \mathbb{P}^{27} of plane sextics). If $\tilde{\phi} : \tilde{Y} \rightarrow \Gamma_6$ is the normalization map, the point $(\Gamma_6, (\tilde{Y}, \tilde{\phi}^*(P_6), \dots, \tilde{\phi}^*(P_{10})))$ lies in $X(F)$. Thus we have that

$$\dim X(F) = \dim m^{-1}(U) \times_U \pi^{-1}(U) - 1 = \dim \pi^{-1}(U) + 9.$$

As m_1 is proper, it follows that $m_1(X(F))$ is closed inside $\pi^{-1}(U)$. Moreover,

$$\dim m_1(X(F)) = \dim X(F) - \dim X_e = \dim \pi^{-1}(U) + 9 - \dim X_e,$$

where X_e is the generic fibre of $m_1|_{X(F)}$. Hence, $\dim m_1(X(F)) < \dim \pi^{-1}(U)$ if and only if $\dim X_e = 10$, that is, the generic fibre of $m_1|_{X(F)}$ coincides with the generic fibre of m_1 . If we prove that this cannot happen, then $m_1|_{X(F)}$ is surjective and in particular $(Y, x, z_1, \dots, z_4) \in m_1(X(F))$, which implies the existence of a plane model $\tilde{\Gamma}$ of Y and of a cubic passing through the singular points of $\tilde{\Gamma}$ and through the images in $\tilde{\Gamma}$ of x, z_1, \dots, z_4 . Therefore it remains only to prove that $\dim X_e \neq 10$.

Given a general $[Y'] \in U$, we have to find general points $z'_1, \dots, z'_5 \in Y'$, a g_6^2 on Y' , together with a frame \mathcal{B}' corresponding to a rational map

$$\phi' : Y' \rightarrow \Gamma' \subset \mathbb{P}^2,$$

such that Γ' has 5 nodes P'_1, \dots, P'_5 and there does not exist a cubic through $P'_1, \dots, P'_5, \phi'(z'_1), \dots, \phi'(z'_5)$. We remark that any complete g_6^2 on Y' is of the form

$$A' = \omega_{Y'} \otimes \mathcal{O}_{Y'}(-a - b), \quad a, b \in Y'.$$

Having chosen a frame for $H^0(Y', A')$ and denoted by $\phi' : Y' \rightarrow \Gamma' \subset \mathbb{P}^2$ the corresponding morphism, this is equivalent to saying that

$$\phi'^* \mathcal{O}_{\Gamma'}(1) = \phi'^*(\mathcal{O}_{\Gamma'}(3)(-\Delta_{\Gamma'})) \otimes \mathcal{O}_{Y'}(-a - b),$$

that is, every cubic in \mathbb{P}^2 passing through the singular points of Γ' and the points $\phi'(a), \phi'(b)$, intersects Γ' in other points which are collinear. Choose \mathcal{B}' any frame of $\omega_{Y'}(-z'_1 - z'_2)$; it is enough to take z'_3, z'_4, z'_5 such that $\phi'(z'_3), \phi'(z'_4), \phi'(z'_5)$ are not collinear in the plane model of Y' corresponding to $(\omega_{Y'} \otimes \mathcal{O}_{Y'}(-z'_1 - z'_2), \mathcal{B}')$.

2 The Gieseker-Petri divisor in M_g for genus $g \leq 13$

Now we consider the case when $g(\Gamma) = 6$, namely Γ is a plane sextic with 4 double points P_1, \dots, P_4 . Using the notation introduced above, we now have that B has degree 2 and so $\eta \in \text{Pic}^{-1}(Y)$. Choose a point $P \in Y$. Since $\text{Pic}^{-2}(Y) = Y_2 - Y_4$, we can always write $\eta - P = D_2 - D_4$; it follows that $\eta = D_3 - D_4$ with P a point of D_3 . As in the previous case, we can assume that

$$h^0(C, \mathcal{O}_C(f^*D_4)) = h^0(Y, \mathcal{O}_Y(D_4)) + h^0(Y, \mathcal{O}_Y(D_3)) = 2,$$

and so $f^*(D_4)$ defines a g_8^1 on C .

In trying to prove that it does not satisfy the Gieseker-Petri Theorem, the above method fails. Indeed, we should prove the existence of a plane cubic passing through $P_1, \dots, P_4, p(z_1), \dots, p(z_6), p(P)$, where $D_4 = z_1 + \dots, z_4$ and $D_3 = z_5 + z_6 + P$. As $P \in Y$ is arbitrarily chosen, it would be enough to prove the existence of a cubic through $P_1, \dots, P_4, z_1, \dots, z_6$ and this is a divisorial condition in $(\mathbb{P}^2)^{10}$. Since $\rho(6, 2, 6) = 0$, in this case we do not have any degree of freedom in the choice of a g_6^2 on Y , namely in the choice of P_1, \dots, P_4 .

Thus we proceed in a slightly different way. We have that $\rho(6, 2, 7) = 3$ and, given l a base point free g_7^2 on Y , we can assume that it defines a birational morphism

$$\varphi : Y \rightarrow \Lambda \subset \mathbb{P}^2,$$

where Λ is a plane septic of genus 6; indeed, l cannot be composed with any involution. We expect Λ to have only nodes as singularities but in this case we cannot exclude the possibility that Λ has some triple points. As Y is the normalization of Λ , we have that

$$\omega_Y = \varphi^*(\mathcal{O}_\Lambda(4)(-\Delta_\Lambda)) \text{ with } \Delta_\Lambda = \sum_{P \in \text{Sing} \Lambda} (r_P - 1)P,$$

where r_P is the multiplicity of Λ in P . Of course for Λ generic, the singular locus Δ_Λ is the sum of the nine nodes P_1, \dots, P_9 and the inequality $\ker \mu_{0, \mathcal{O}_C(f^*D_4)} \neq 0$ is equivalent to the existence of a plane quartic through the points $P_1, \dots, P_9, \varphi(z_1), \dots, \varphi(z_6)$. In the non generic case the condition equivalent to the non-injectivity of $\mu_{0, \mathcal{O}_C(f^*D_4)}$ is different (for instance, when Λ has a triple point Q and six double points P_1, \dots, P_6 , then we require that the plane quartic has a double point in Q and passes through P_1, \dots, P_6). However, the number of independent conditions imposed on the plane quartics is the same.

As before, we consider a neighborhood U of Y in M_6 such that there exists a proper morphism $\xi : \mathcal{G}_7^2 \rightarrow U$, where \mathcal{G}_7^2 parametrizes pairs (Y', l') , with $[Y'] \in U$ and l' a g_7^2 on Y' . We can assume that, given $[Y'] \in U$, the generic g_7^2 on Y' is base point free and not composed with an involution but in this case the models of Y' as a plane septic can have also some triple points. Denoting by $m : V_{6,7} \rightarrow \mathcal{M}_6$ the natural morphism, the restriction $m_U : m^{-1}(U) \rightarrow U$ is proper. If $\pi : M_{6,6} \rightarrow M_6$ is the forgetful map, then the

morphism $m_1 : m^{-1}(U) \times_U \pi^{-1}(U) \rightarrow \pi^{-1}(U)$ is proper, too. We have that

$$\begin{aligned} \dim m^{-1}(U) \times_U \pi^{-1}(U) &= \dim \pi^{-1}(U) + \rho(6, 2, 7) + \dim PGL(2) = \\ &= \dim \pi^{-1}(U) + 11. \end{aligned}$$

As in the previous case, we define

$$\mathcal{E} := H^0(\mathcal{O}_{\mathbb{P}^2}(4)) \times (m^{-1}(U) \times_U \pi^{-1}(U))$$

and \mathcal{F} being the bundle on $m^{-1}(U) \times_U \pi^{-1}(U)$ with fibre over $(\Lambda', (Y', z'_1, \dots, z'_6))$ equal to

$$H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\Delta_{\Lambda'}}) \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\varphi'(z'_1)}) \oplus \dots \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\varphi'(z'_6)}),$$

where $\varphi' : Y' \rightarrow \Lambda'$ is the normalization map. For $\Lambda' \in m^{-1}(U)$ generic we have that

$$H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\Delta_{\Lambda'}}) = H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{P'_1}) \oplus \dots \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{P'_9}),$$

where P'_1, \dots, P'_9 are the nodes of Λ' . Instead, if for instance Λ' has one triple point Q' and 6 nodes P'_1, \dots, P'_6 , then the following equality holds:

$$H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\Delta_{\Lambda'}}) = H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{2Q'}) \oplus \dots \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{P'_6}).$$

We define $F : \mathcal{E} \rightarrow \mathcal{F}$ to be the evaluation map. As both \mathcal{E} and \mathcal{F} have rank 15, the situation is analogous to the one already treated. Therefore, in order to prove that the image under m_1 of the degeneracy locus $X(F)$ is the whole $\pi^{-1}(U)$, it is enough to show that the generic fibre X_e of $m_1|_{X(F)}$ is nonempty and that it does not coincide with the generic fibre of m_1 .

The fact that $X_e \neq \emptyset$ follows easily by observing that, given 15 points on a quartic $\Lambda_4 \subset \mathbb{P}^2$, there always exists a plane septic Λ_7 passing through them and having nodes in the first nine.

On the other hand, it can be shown that $\dim X_e \neq 15$ by proceeding like in the case of genus 5 because on a curve Y' of genus 6 any complete g_7^2 has to be of the form $l' = \omega_{Y'} \otimes \mathcal{O}_{Y'}(-a - b - c)$, with $a, b, c \in Y'$. \square

Finally, we obtain that:

Corollary 2.5.6. *The locus GP_{12} has pure codimension 1 in M_{12} .*

Proof. By the remarks at the beginning of the section we have to study only the component $GP_{12,10}^1$.

Given $[C] \in GP_{12,10}^1$, we may assume that $[C]$ does not lie in $GP_{12,9}^1 \cup GP_{12,10}^2 \cup GP_{12,11}^2$, which forces any $l \in G_{10}^1(C)$ for which the Gieseker-Petri Theorem fails to verify condition 1. By Theorem 2.5.5, condition 2 is satisfied if $[C] \notin GP_{12,7}^1 \cup GP_{12,8}^1$. We can thus apply Proposition 2.5.1. \square

Bibliography

- [1] E. Arbarello and M. Cornalba. Footnotes to a paper of Beniamino Segre. The number of g_1^1 's on a general d -gonal curve, and the unirationality of the Hurwitz spaces of 4-gonal and 5-gonal curves. *Math. Ann.*, 256:341–362, 1981.
- [2] E. Arbarello and M. Cornalba. A few remarks about the variety of irreducible plane curves of given degree and genus. *Ann. Sci. École Norm. Sup. (4)*, 16(3):467–488 (1984), 1983.
- [3] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris. *Geometry of algebraic curves. Volume I*, volume 267 of *Grundle Math. Wiss.*. Springer Verlag, 1985.
- [4] A. Bruno and E. Sernesi. A note on the Petri loci. *Manuscr. Math.*, 136(3-4):439–443, 2011.
- [5] A. Castorena. Curves of genus seven that do not satisfy the Gieseker–Petri theorem. *Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8)*, 8(3):697–706, 2005.
- [6] A. Castorena. A family of plane curves with moduli $3g - 4$. *Glasg. Math. J.*, 49(3):417–422, 2007.
- [7] A. Castorena. Remarks on the Gieseker-Petri divisor in genus eight. *Rend. Circ. Mat. Palermo (2)*, 59(1):143–150, 2010.
- [8] D. Edidin. Brill-Noether theory in codimension-two. *J. Algebr. Geom.*, 2(1):25–67, 1993.
- [9] D. Eisenbud and J. Harris. A simpler proof of the Gieseker-Petri theorem on special divisors. *Invent. Math.*, 74:269–280, 1983.
- [10] D. Eisenbud and J. Harris. The Kodaira dimension of the moduli space of curves of genus ≥ 23 . *Invent. Math.*, 90:359–387, 1987.
- [11] D. Eisenbud and J. Harris. Irreducibility of some families of linear series with Brill-Noether number -1 . *Ann. Sci. Éc. Norm. Supér. (4)*, 22(1):33–53, 1989.
- [12] D. Eisenbud, H. Lange, G. Martens, and F.-O. Schreyer. The Clifford dimension of a projective curve. *Compos. Math.*, 72(2):173–204, 1989.
- [13] G. Farkas. Gaussian maps, Gieseker-Petri loci and large theta-characteristics. *J. Reine Angew. Math.*, 581:151–173, 2005.

Bibliography

- [14] G. Farkas. Rational maps between moduli spaces of curves and Gieseker-Petri divisors. *J. Algebr. Geom.*, 19(2):243–284, 2010.
- [15] W. Fulton. Hurwitz schemes and irreducibility of moduli of algebraic curves. *Ann. Math. (2)*, 90:542–575, 1969.
- [16] D. Gieseker. Stable curves and special divisors: Petri’s conjecture. *Invent. Math.*, 66:251–275, 1982.
- [17] J. Harris. Theta-characteristics on algebraic curves. *Trans. Am. Math. Soc.*, 271:611–638, 1982.
- [18] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [19] G. Martens. Über den Clifford-Index algebraischer Kurven. *J. Reine Angew. Math.*, 336:83–90, 1982.
- [20] D. Mumford. Theta characteristics of an algebraic curve. *Ann. Sci. Éc. Norm. Supér. (4)*, 4:181–192, 1971.
- [21] B. Segre. Sui moduli delle curve poligonali, e sopra un complemento al teorema di esistenza di Reimann. *Math. Ann.*, 100(1):537–551, 1928.
- [22] F. Steffen. A generalized principal ideal theorem with an application to Brill-Noether theory. *Invent. Math.*, 132(1):73–89, 1998.
- [23] M. Teixidor i Bigas. Half-canonical series on algebraic curves. *Trans. Am. Math. Soc.*, 302:99–115, 1987.
- [24] M. Teixidor i Bigas. The divisor of curves with a vanishing theta-null. *Compos. Math.*, 66(1):15–22, 1988.

3 Stability of rank-3 Lazarsfeld-Mukai bundles on $K3$ surfaces

3.1 Introduction and statement of the results

Many results of Brill-Noether theory regarding a general point in the moduli space M_g , which parametrizes isomorphism classes of smooth, irreducible curves of genus g , have been proved by studying curves lying on $K3$ surfaces. One of the advantages of considering an irreducible curve $C \subset S$, where S is a smooth $K3$ surface, is that some interesting properties, such as the Clifford index, do not change while moving C in its linear system (cf. [11]); we refer to Sections 1.5 and 1.6 for details. Moreover, Brill-Noether theory on C is strictly connected with the geometry of some moduli spaces of vector bundles on the $K3$ surface. Indeed, given a complete, base point free linear series A on C , one associates with the pair (C, A) a vector bundle on S , the so-called Lazarsfeld-Mukai bundle, denoted by $E_{C,A}$.

As recalled in Section 1.4, Lazarsfeld-Mukai bundles were first used by Lazarsfeld, in order to show that, given a $K3$ surface S such that $\text{Pic}(S) = \mathbb{Z} \cdot L$, a general curve $C \in |L|$ satisfies the Gieseker-Petri Theorem, that is, for any line bundle $A \in \text{Pic}(C)$ the Petri map

$$\mu_{0,A} : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee) \rightarrow H^0(C, \omega_C)$$

is injective (cf. [16], [23], or [17] for a more geometric argument).

It is natural to investigate what happens if the Picard number of S is greater than 1. In order to do so, having denoted by $|L|_s$ the locus of smooth, connected curves in the linear system $|L|$ and chosen two positive integers r, d , one studies the natural projection $\pi : \mathcal{W}_d^r(|L|) \rightarrow |L|_s$, whose fibre over C coincides with the Brill-Noether variety $W_d^r(C)$. We set $g := 1 + c_1(L)^2/2$; this coincides with the genus of curves in $|L|_s$.

At first we look at the cases where $\rho(g, r, d) < 0$. Following [5], we say that a line bundle M is *adapted* to $|L|$ whenever

$$(i) \ h^0(S, M) \geq 2, h^0(S, L \otimes M^\vee) \geq 2,$$

$$(ii) \ h^0(C, M \otimes \mathcal{O}_C) \text{ is independent of the curve } C \in |L|_s.$$

Conditions (i) and (ii) ensure that $M \otimes \mathcal{O}_C$ contributes to the Clifford index of C and $\text{Cliff}(M \otimes \mathcal{O}_C)$ is the same for any $C \in |L|_s$.

Donagi and Morrison ([5, Theorem (5.1')]) proved that, if A is a complete, base point free pencil g_d^1 on a nonhyperelliptic curve $C \in |L|_s$ and $\rho(g, 1, d) < 0$, then $|A|$ is contained in the restriction to C of a line bundle $M \in \text{Pic}(S)$ which is adapted to $|L|$ and such that $\text{Cliff}(M \otimes \mathcal{O}_C) \leq \text{Cliff}(A)$. The same is expected to hold true for any linear

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

series of type g_d^r with $\rho(g, r, d) < 0$ (compare with [5, Conjecture (1.2)]). We prove this conjecture for $r = 2$ under some mild hypotheses on L .

Theorem 3.1.1. *Let S be a K3 surface and $L \in \text{Pic}(S)$ be an ample line bundle such that a general curve in $|L|$ has genus g , Clifford dimension 1 and maximal gonality $k = \lfloor \frac{g+3}{2} \rfloor$. Let A be a complete, base point free g_d^2 on a curve $C \in |L|_s$ such that $\rho(g, 2, d) < 0$.*

Then, there exists $M \in \text{Pic}(S)$ adapted to $|L|$ such that the linear system $|A|$ is contained in $|M \otimes \mathcal{O}_C|$ and $\text{Cliff}(M \otimes \mathcal{O}_C) \leq \text{Cliff}(A)$. Moreover, one has $c_1(M) \cdot C \leq (4g - 4)/3$.

We recall that the assertion that $|A|$ is contained in $|M \otimes \mathcal{O}_C|$ is equivalent to the requirement $h^0(C, A^\vee \otimes M \otimes \mathcal{O}_C) > 0$. The assumption on the gonality k is used for computational reasons; however, the methods of our proof might be adapted in order to treat the cases where k is not maximal. It was proved by Ciliberto and Pareschi (cf. [4] Proposition 3.3) that the ampleness of $L = \mathcal{O}_S(C)$ forces C to have Clifford dimension 1 with only one exception occurring for $g = 10$.

The case of pencils is very special, since it involves vector bundles of rank 2. Donagi and Morrison used the fact that any non-simple, indecomposable Lazarsfeld-Mukai bundle of rank 2 can be expressed as extension of the image and the kernel of a nilpotent endomorphism, which both have rank 1. Their proof cannot be adapted to linear series with $r > 1$, corresponding to Lazarsfeld-Mukai bundles of rank at least 3. Our techniques consist of showing that, under the hypotheses of Theorem 3.1.1, the rank-3 Lazarsfeld-Mukai bundle $E = E_{C,A}$ is given by an extension

$$0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0,$$

where $N \in \text{Pic}(S)$ and E/N is a μ_L -stable, torsion free sheaf of rank 2. When E is μ_L -unstable, the line bundle N coincides with its maximal destabilizing sheaf and the determinant of E/N plays the role of the line bundle M in the statement. Something similar happens if E is properly μ_L -semistable.

This suggests that the notion of stability might play a fundamental role in a general proof of the Donagi-Morrison Conjecture.

Now, we turn our attention to the cases where $\rho(g, r, d) \geq 0$. In the course of their proof of Green's Conjecture for curves on arbitrary K3 surfaces, Aprodu and Farkas (cf. [1]) showed that, if L is an ample line bundle on a K3 surface such that a general curve $C \in |L|$ has Clifford dimension 1 and gonality k , given $d > g - k + 2$, any dominating component of $\mathcal{W}_d^1(|L|)$ corresponds to simple Lazarsfeld-Mukai bundles. In particular, when the gonality is maximal this ensures (cf. Theorem 1.7.3) that, if C is general in its linear system and the Brill-Noether number $\rho(g, 1, d)$ is positive, the variety $W_d^1(C)$ is reduced and of the expected dimension. In the case $\rho(g, 1, d) = 0$, one finds that $W_d^1(C)$ is 0-dimensional, even though not necessarily reduced.

It is natural to wonder to what extent such a result can be expected to hold for linear series of type g_d^r with $r > 1$. We prove the following theorem.

Theorem 3.1.2. *Let S be a K3 surface and $L \in \text{Pic}(S)$ be an ample line bundle such that a general curve in $|L|$ has genus g , Clifford dimension 1 and maximal gonality $k = \lfloor \frac{g+3}{2} \rfloor$. Fix a positive integer d such that $\rho(g, 2, d) \geq 0$ and assume $(g, d) \notin \{(2, 4), (4, 5), (6, 6), (10, 9)\}$. Then, the following hold:*

- (a) *If $d > \frac{3}{4}g + 2$, no dominating component of $\mathcal{W}_d^2(|L|)$ corresponds to rank-3 Lazarsfeld-Mukai bundles which are not μ_L -stable.*
- (b) *If $d \leq \frac{3}{4}g + 2$, let \mathcal{W} be a dominating component of $\mathcal{W}_d^2(|L|)$ that corresponds to Lazarsfeld-Mukai bundles which are not μ_L -stable. Then, there exists $M \in \text{Pic}(S)$ adapted to $|L|$ such that, for a general $(C, A) \in \mathcal{W}$, the linear system $|A|$ is contained in $|M \otimes \mathcal{O}_C|$ and $\text{Cliff}(M \otimes \mathcal{O}_C) \leq \text{Cliff}(A)$. Moreover, $c_1(M) \cdot C \leq (4g - 4)/3$.*

Unlike case (a), case (b) does not exclude the existence of dominating components of $\mathcal{W}_d^2(|L|)$ which correspond to either μ_L -stable or properly μ_L -semistable Lazarsfeld-Mukai bundles. However, general points of such a component \mathcal{W} give nets g_d^2 , which are all contained in the restriction of the same line bundle $M \in \text{Pic}(S)$ to curves in $|L|$. Furthermore, the Clifford index of $M \otimes \mathcal{O}_C$ is the same for any $C \in |L|_s$ and does not exceed $d - 4$.

For a curve $C \in |L|_s$ and for a fixed value of d , we define the variety

$$\tilde{W}_d^2(C) := \{A \in W_d^2(C) \mid A \text{ is base point free}\},$$

which is an open subscheme of $W_d^2(C)$, not necessarily dense. The following result is a direct consequence of Theorem 3.1.2.

Corollary 3.1.3. *Under the same hypotheses of Theorem 3.1.2, for a general $C \in |L|_s$ the following hold.*

- (a) *If $d > \frac{3}{4}g + 2$, the variety $\tilde{W}_d^2(C)$ is reduced of the expected dimension $\rho(g, 2, d)$.*
- (b) *If $d \leq \frac{3}{4}g + 2$, let W be an irreducible component of $\tilde{W}_d^2(C)$ which either is non-reduced or has dimension greater than $\rho(g, 2, d)$. Then, there exists an effective divisor $D \subset S$ such that $\mathcal{O}_S(D)$ is adapted to $|L|$ and, for a general $A \in W$, the linear system $|A|$ is contained in $|\mathcal{O}_C(D)|$ and $\text{Cliff}(\mathcal{O}_C(D)) \leq \text{Cliff}(A)$.*

Aprodu and Farkas' result follows from a parameter count for spaces of Donagi-Morrison extensions corresponding to non-simple Lazarsfeld-Mukai bundles of rank 2. The strategy used to prove Theorem 3.1.2 consists, instead, of counting the number of moduli of μ_L -unstable and properly μ_L -semistable Lazarsfeld-Mukai bundles of rank 3; this involves Artin stacks that parametrize the corresponding Harder-Narasimhan and Jordan-Hölder filtrations.

The plan of the paper is as follows. Sections 3.2, 3.3 and 3.4 give background information on linear systems on K3 surfaces, Lazarsfeld-Mukai bundles and stability of sheaves on K3 surfaces respectively.

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

In Section 3.5 we present a different proof of Aprodu and Farkas' result and show that, if $\rho(g, 1, d) > 0$, the Lazarsfeld-Mukai bundles corresponding to general points of any dominating component of $\mathcal{W}_d^1(|L|)$ are not only simple, but even μ_L -stable (Theorem 3.5.3). We introduce stacks of filtrations, studied for instance by Bridgeland in [2] and Yoshioka in [26], and explain our parameter count in an easier case. The space of Lazarsfeld-Mukai bundles E , such that the bundles appearing in the Harder-Narasimhan filtration of E have prescribed Mukai vectors, turns out to be an Artin stack, whose dimension can be computed by using some well known facts regarding morphisms between semistable sheaves.

In Section 3.6 we look at the different types of possible Harder-Narasimhan and Jordan-Hölder filtrations of a rank-3 Lazarsfeld-Mukai bundle E with $\det(E) = L$ and $c_2(E) = d$. If the determinants of both the subbundles E_i and the quotient sheaves E^j , given by the filtration of E , have at least two global sections, their restriction to a general curve $C \in |L|$ contributes to the Clifford index. This is used in order to bound from below the intersection products between the first Chern classes of the sheaves E_i and E^j .

In Sections 3.7, 3.8, 3.9 we estimate the number of moduli of pairs (C, A) corresponding to rank-3 Lazarsfeld-Mukai bundles which are not μ_L -stable. The subdivision in three sections reflects the different methods necessary to treat various types of filtrations, depending on their length and on the rank of the sheaves E_i and E^j . At the end of Section 3.9 the proofs of Theorems 3.1.1 and 3.1.2 are given. In Section 3.10, an application towards transversality of Brill-Noether loci and Gieseker-Petri loci is presented. For values of r, d such that $\rho(g, r, d) \geq 0$, we define the scheme

$${}^cGP_{g,d}^r := \{[C] \in M_g \mid \exists A \in W_d^r(C) \setminus W_d^{r+1}(C) \text{ with } \ker \mu_{0,A} \neq 0\},$$

which is open but not necessarily dense in the locus $GP_{g,d}^r$. We prove the following:

Theorem 3.1.4. *Let $r \geq 3$, $g \geq 0$, $d \leq g - 1$ be positive integers such that $\rho(g, r, d) < 0$ and $d - 2r + 2 \geq \lfloor (g + 3)/2 \rfloor$. If $r \geq 4$, assume $d^2 > 4(r - 1)(g + r - 2)$. For $r = 3$, let $d^2 > 8g + 1$. If -1 is not represented by the quadratic form*

$$Q(m, n) = (r - 1)m^2 + mnd + (g - 1)n^2, \quad m, n \in \mathbb{Z},$$

then:

- a. $M_{g,d}^r \not\subset M_{g,f}^1$ for $f < (g + 2)/2$.
- b. $M_{g,d}^r \not\subset {}^cGP_{g,f}^1$ for $f \geq (g + 2)/2$.
- c. $M_{g,d}^r \not\subset M_{g,e}^2$ if $e < d - 2r + 5$ and $\rho(g, 2, e) < 0$.
- d. $M_{g,d}^r \not\subset {}^cGP_{g,e}^2$ if $e < \min \{ \frac{17}{24}g + \frac{23}{12}, d - 2r + 5 \}$ and $\rho(g, 2, e) \geq 0$.

The assumption on the quadratic form Q is a mild hypothesis. For instance, it is automatically satisfied when r and g are odd and d is even.

In the last section we exhibit an application of our methods to higher rank Brill-Noether Theory. We give a negative answer to Question 4.2 in [13], which asks whether the second Clifford index $\text{Cliff}_2(C)$, associated with rank-2 vector bundles on an algebraic curve C , equals $\text{Cliff}(C)$ whenever C is a Petri curve. We analyze what happens in genus 11 and look at the Noether-Lefschetz divisor $\mathcal{NL}_{11,13}^4$, which consists of curves that lie on a K3 surface $S \subset \mathbb{P}^4$ with Picard number at least 2; this coincides with the locus of curves $[C] \in M_{11}$ such that $\text{Cliff}_2(C) < \text{Cliff}(C)$ (cf. [7]). We prove the following:

Theorem 3.1.5. *A general curve $[C] \in \mathcal{NL}_{11,13}^4$ satisfies the Gieseker-Petri Theorem.*

In other words, the Gieseker-Petri divisor GP_{11} and the Noether-Lefschetz divisor $\mathcal{NL}_{11,13}^4$ are transversal.

3.2 Linear systems on K3 surfaces

The following results concerning linear systems on K3 surfaces are mainly due to Mayer ([18]) and Saint-Donat ([24]).

Let D be a divisor on a K3 surface S and set $L := \mathcal{O}_S(D)$. Riemann-Roch Theorem and Serre duality give

$$h^0(S, L) + h^0(S, L^{-1}) = 2 + \frac{1}{2}D^2 + h^1(S, L).$$

It follows that, if L is numerically equivalent to 0, then either L or L^{-1} has some sections and neither D nor $-D$ is effective and non-trivial; hence, $L = \mathcal{O}_S$. As a consequence, the Picard group of S has no torsion and $\text{Pic}(S) = \text{Num}(S)$.

The short exact sequence defining \mathcal{O}_D trivially implies that

$$h^1(S, \mathcal{O}_S(-D)) = h^0(D, \mathcal{O}_D) - 1.$$

Therefore, if D is an irreducible curve, then $\dim |L| = 1 + D^2/2$ and, having denoted by $p_a(D)$ its arithmetic genus, one has $D^2 = 2p_a(D) - 2$ by the adjunction formula. In particular, one finds $D^2 \geq -2$ and

$$\begin{aligned} \dim |L| = 0 &\iff D^2 = -2 &\iff D \text{ is smooth and rational,} \\ \dim |L| = 1 &\iff D^2 = 0 &\iff p_a(D) = 1. \end{aligned}$$

In general, if L is any line bundle on S such that $h^0(S, L) > 0$, then any fixed component of $|L|$ is a (-2) -curve which is isomorphic to \mathbb{P}^1 . Moreover, $|L|$ has no base points outside its fixed components (cf. [24, Corollary 3.2]).

The following result is a strong version of Bertini's Theorem holding for K3 surfaces.

Theorem 3.2.1. *Let L be a line bundle on a K3 surface S such as $h^0(S, L) > 0$ and $|L|$ has no fixed components. Then, the following hold:*

- (i) *If $c_1(L)^2 > 0$, then $h^1(S, L) = 0$ and a general element in $|L|$ is a smooth, irreducible curve of genus $g = 1 + c_1(L)^2/2$.*

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

- (ii) If $c_1(L)^2 = 0$, then there exist a number $k \in \mathbb{Z}^{>0}$ and an irreducible curve $E \subset S$ with $p_a(E) = 1$ such that $L = \mathcal{O}_S(kE)$. In this case, one obtains $h^0(S, L) = k + 1$, $h^1(S, L) = k - 1$ and every element in $|L|$ can be written as a sum $E_1 + E_2 + \dots + E_k$ with $E_i \in |E|$ for $1 \leq i \leq k$.

In the first case, we call any smooth curve $C \in |L|$ a *K3-section*. Let $c_1(L)^2 > 0$ and look at the map $\phi_L : S \rightarrow \mathbb{P}(H^0(S, L)) \simeq \mathbb{P}^8$. Two cases can occur. Either ϕ_L is birational and contracts only the finitely many (-2) -curves $\Gamma \subset S$ such that $\Gamma \cdot L = 0$, or it has degree 2 and every member of $|L|$ is a hyperelliptic curve; in the latter case, the map ϕ_{L^3} is birational. Notice that the restriction of ϕ_L to a smooth curve $C \in |L|$ is the canonical morphism ϕ_{ω_C} .

3.3 Lazarsfeld-Mukai bundles

In this section we briefly recall the definition and the main properties of Lazarsfeld-Mukai (henceforth LM) bundles associated with complete, base point free linear series on curves lying on K3 surfaces. We refer to Section 1.4 for details. Let S be a K3 surface and $C \subset S$ a smooth connected curve of genus g . Any base point free linear series $A \in W_d^r(C) \setminus W_d^{r+1}(C)$ can be considered as a globally generated sheaf on S ; therefore, the evaluation map $\text{ev}_{A,S} : H^0(C, A) \otimes \mathcal{O}_S \rightarrow A$ is surjective and one defines the bundle $F_{C,A}$ to be its kernel, i.e.,

$$0 \rightarrow F_{C,A} \rightarrow H^0(C, A) \otimes \mathcal{O}_S \rightarrow A \rightarrow 0. \quad (3.1)$$

The LM bundle associated with the pair (C, A) is, by definition, $E_{C,A} := F_{C,A}^\vee$. By dualizing (3.1), one finds that $E_{C,A}$ sits in the following short exact sequence:

$$0 \rightarrow H^0(C, A)^\vee \otimes \mathcal{O}_S \rightarrow E_{C,A} \rightarrow \omega_C \otimes A^\vee \rightarrow 0; \quad (3.2)$$

in particular, $E_{C,A}$ is equipped with a $(r + 1)$ -dimensional subspace of sections. The following proposition summarizes the most important properties of $E_{C,A}$:

Proposition 3.3.1. *If $E_{C,A}$ is the LM bundle corresponding to a base point free linear series $A \in W_d^r(C) \setminus W_d^{r+1}(C)$, then:*

- $\text{rk } E_{C,A} = r + 1$.
- $\det E_{C,A} = L$, where $C \in |L|$.
- $c_2(E_{C,A}) = d$.
- The bundle $E_{C,A}$ is globally generated off the base locus of $\omega_C \otimes A^\vee$.
- $h^0(S, E_{C,A}) = h^0(C, A) + h^0(C, \omega_C \otimes A^\vee) = r + 1 + g - d + r$,
 $h^1(S, E_{C,A}) = h^2(S, E_{C,A}) = 0$.
- $\chi(S, E_{C,A} \otimes F_{C,A}) = 2(1 - \rho(g, r, d))$.

3.4 Mumford stability for sheaves on K3 surfaces

In particular, if $\rho(g, r, d) < 0$, the LM bundle $E_{C,A}$ is non-simple. Being a LM bundle is an open condition. Indeed, a vector bundle E of rank $r + 1$ is a LM bundle whenever $h^1(S, E) = h^2(S, E) = 0$ and there exists a subspace Λ of $G(r + 1, H^0(S, E))$ such that the degeneracy locus of the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \rightarrow E$ is a smooth connected curve.

We recall Proposition 1.4.2 due to Pareschi, stating that, if $C \in |L|_s$ is general, for any base point free $A \in \text{Pic}(C)$ one has:

$$\ker \mu_{0,A} = 0 \iff E_{C,A} \text{ is simple.}$$

Standard Brill-Noether theory implies that no component of $\mathcal{W}_d^r(|L|)$ is entirely contained in $\mathcal{W}_d^{r+1}(|L|)$. Therefore, the variety $\mathcal{W}_d^r(C)$ is reduced of the expected dimension for a general $C \in |L|_s$ as soon as no dominating component \mathcal{W} of $\mathcal{W}_d^r(|L|)$ is of one of the following types:

- (a) For $(C, A) \in \mathcal{W}$ general, A is complete, base point free and $E_{C,A}$ is non-simple.
- (b) For $(C, A) \in \mathcal{W}$ general, A is not base point free and $\ker \mu_{0,A} \neq 0$.

In order to exclude (b), one can proceed by induction on d because, if B denotes the base locus of A and $\ker \mu_{0,A} \neq 0$, then $\mu_{0,A(-B)} \neq 0$, too.

3.4 Mumford stability for sheaves on K3 surfaces

For later use, we recall some facts about coherent sheaves on smooth projective surfaces, referring to [12] and [25] for most of the proofs. Let S be a smooth, projective surface over \mathbb{C} and H an ample line bundle on it. Given a torsion free sheaf E on S of rank r , the H -slope of E is defined as

$$\mu_H(E) = \frac{c_1(E) \cdot c_1(H)}{r};$$

E is called μ_H -semistable (resp. μ_H -stable) in the sense of Mumford-Takemoto if for any subsheaf $0 \neq F \subset E$ with $\text{rk } F < \text{rk } E$, one has $\mu_H(F) \leq \mu_H(E)$ (resp. $\mu_H(F) < \mu_H(E)$). The Harder-Narasimhan (henceforth HN) filtration of E is the unique filtration

$$0 = E_0 \subset E_1 \subset \dots \subset E_s = E,$$

such that $E^i := E_i/E_{i-1}$ is a torsion free, μ_H -semistable sheaf for $1 \leq i \leq s$, and $\mu_H(E_{i+1}/E_i) < \mu_H(E_i/E_{i-1})$ for $1 \leq i \leq s - 1$. Such a filtration always exists. It can be easily checked that, if E is a vector bundle, the sheaves E_i are locally free; moreover,

$$\mu_H(E_1) > \mu_H(E_2) > \dots > \mu_H(E_s).$$

The sheaf E_1 is called the maximal destabilizing sheaf of E ; the number $\mu_H(E_1)$ is the maximal slope of a proper subsheaf of E and, among the subsheaves of E of slope equal to $\mu_H(E_1)$, the sheaf E_1 has maximal rank. In particular, E_1 is μ_H -semistable.

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

Now, we assume E is μ_H -semistable. A Jordan-Hölder filtration of E (later on, JH filtration) is a filtration

$$0 = JH_0(E) \subset JH_1(E) \subset \dots \subset JH_s(E) = E,$$

such that all the factors $\text{gr}_i(E) := JH_i(E)/JH_{i-1}(E)$ are torsion free, μ_H -stable sheaves of slope equal to $\mu_H(E)$. This implies that $\mu_H(JH_i(E)) = \mu_H(E)$ for $1 \leq i \leq s$. The Jordan-Hölder filtration always exists but is not uniquely determined, while the graded object $\text{gr}(E) := \bigoplus_i \text{gr}_i(E)$ is. The following result regards morphisms between μ_H -semistable and μ_H -stable sheaves on S (cf. [25], [9]).

Proposition 3.4.1. *Let E, F be torsion free sheaves on S . Then:*

- a. *If E and F are μ_H -semistable and $\mu_H(E) > \mu_H(F)$, then $\text{Hom}(E, F) = 0$.*
- b. *If E and F are μ_H -stable, $\mu_H(E) = \mu_H(F)$ and there exists a nonzero $\varphi \in \text{Hom}(E, F)$, then $\text{rk } E = \text{rk } F$ and φ is an isomorphism in codim ≤ 1 (in particular it is injective).*

In the case where S is a K3 surface, by Serre duality $H^2(S, E) \simeq \text{Hom}(E, \mathcal{O}_S)^\vee$; hence (a) implies that, if E is μ_H -semistable and $\mu_H(E) > 0$, then $h^2(S, E) = 0$.

From now on, we assume S to be a K3 surface. Throughout the paper we will often use the following fact:

Lemma 3.4.2. *Let $E, Q \in \text{Coh}(S)$ be torsion free and $\text{rk } E \geq 2$. If E is globally generated off a finite number of points, $h^2(S, E) = 0$ and there exists a surjective morphism $\varphi : E \rightarrow Q$, then $h^0(S, Q^{\vee\vee}) \geq 2$.*

Proof. Being a quotient of E , the sheaf Q is globally generated off a finite set. If $\text{rk } Q \geq 2$, this trivially implies $h^0(S, Q^{\vee\vee}) \geq h^0(S, Q) \geq 2$. On the other hand, if Q has rank 1, then $Q = N \otimes I$, where $N \in \text{Pic}(S)$ and I is the ideal sheaf associated with a 0-dimensional subscheme of S . Since N is a quotient of E off a finite number of points, it has no fixed components, thus it is base point free (cf. [24]). The statement follows by remarking that $N = Q^{\vee\vee}$ cannot be trivial because $h^2(S, E) = 0$. \square

Another useful result is the following one (cf. Lemma 3.1 in [11]):

Lemma 3.4.3. *Let E be a vector bundle of rank r on S which is globally generated off a finite number of points. If $h^2(S, E) = 0$, then $h^0(S, \det E) \geq 2$.*

Proof. Since the natural map $\wedge^r H^0(S, E) \otimes \mathcal{O}_S \rightarrow \wedge^r E = \det E$ is surjective off a finite number of points, the line bundle $\det E$ is base point free. Therefore, it is enough to show that $\det E$ is non-trivial. This follows by remarking that, if $V \in G(r, H^0(S, E))$ is general, then the natural map $ev_V : V \otimes \mathcal{O}_S \rightarrow E$ is injective but is not an isomorphism since $h^2(S, E) = 0$. Therefore, $\det ev_V$ gives a section of $\det E$ vanishing on a non-zero effective divisor. \square

3.5 Stability of Lazarsfeld-Mukai bundles of rank 2

Last but not least, we recall some notation and results from [20]. The Mukai vector of a sheaf $E \in \text{Coh}(S)$ is defined as:

$$v(E) := \text{ch}(E)(1 + \omega) = \text{rk}(E) + c_1(E) + (\chi(E) - \text{rk}(E))\omega \in H^*(S, \mathbb{Z}) = H^{2*}(S, \mathbb{Z}),$$

where $H^4(S, \mathbb{Z})$ is identified with \mathbb{Z} by means of the fundamental cocycle ω . The Chern classes of E are fixed once its Mukai vector is assigned.

The Mukai lattice is the pair $(H^*(S, \mathbb{Z}), \langle, \rangle)$, with \langle, \rangle being the symmetric bilinear form on $H^*(S, \mathbb{Z})$ whose definition is the following:

$$\langle v, w \rangle := - \int_S v^* \wedge w,$$

where, if $v = v^0 + v^1 + v^2$ with $v^i \in H^{2i}(S, \mathbb{Z})$, we set $v^* := v^0 - v^1 + v^2$. Given $E, F \in \text{Coh}(S)$, we define the Euler characteristic of the pair (E, F) as

$$\chi(E, F) := \sum_{i=0}^2 (-1)^i \dim \text{Ext}^i(E, F),$$

and it turns out that $\chi(E, F) = -\langle v(E), v(F) \rangle$.

Given a Mukai vector $v \in H^*(S, \mathbb{Z})$, let $\mathcal{M}(v)$ be the moduli stack of coherent sheaves on S of Mukai vector v . If $H \in \text{Pic}(S)$ is ample, we denote by $\mathcal{M}_H(v)^{\mu_{ss}}$ (resp. $\mathcal{M}_H(v)^{\mu_s}$) the moduli stack parametrizing isomorphism classes of μ_H -semistable (resp. μ_H -stable) sheaves on S with Mukai vector v . Recall that any μ_H -stable sheaf is simple and that any irreducible component of $\mathcal{M}_H(v)^{\mu_s}$ has dimension equal to $\langle v, v \rangle + 1$. Moreover, if $\gcd(v^0, v^1 \cdot H) = 1$, then μ_H -semistability and μ_H -stability coincide.

3.5 Stability of Lazarsfeld-Mukai bundles of rank 2

Let S be a smooth, projective $K3$ surface and consider an ample line bundle $L \in \text{Pic}(S)$ such that a general curve $C \in |L|_S$ has genus g , Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$. In this section we prove that, if C is general in its linear system and $\rho(g, 1, d) > 0$, the LM bundle associated with a general complete, base point free g_d^1 on C is μ_L -stable.

Fix a rank-2 LM bundle $E = E_{C,A}$ corresponding to a complete, base point free pencil $A \in W_d^1(C)$ with $C \in |L|_S$; Proposition 3.3.1 implies that

$$v(E) = 2 + c_1(L) + (g - d + 1)\omega.$$

We assume E is not μ_L -stable. In the case where E is μ_L -unstable (respectively properly μ_L -semistable) we consider its HN filtration (resp. JH filtration) $0 \subset M \subset E$, which gives a short exact sequence

$$0 \rightarrow M \rightarrow E \rightarrow N \otimes I_{\xi} \rightarrow 0, \tag{3.3}$$

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

where M and N are two line bundles such that $\mu_L(M) > \mu_L(E) = g - 1 > \mu_L(N)$ (resp. $\mu_L(M) = \mu_L(E) = \mu_L(N)$) and I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$ of length $l = d - c_1(N) \cdot c_1(M)$. By Lemma 3.4.2, we know that $h^0(S, N) \geq 2$. First of all, we prove the following:

Lemma 3.5.1. *In the situation above, if general curves in $|L|_s$ have Clifford dimension 1 and (constant) gonality k , one has $c_1(M) \cdot c_1(N) \geq k$.*

Proof. We remark that $h^2(S, M) = 0$ since $\mu_L(M) > 0$. Therefore, if

$$2 > h^0(S, M) \geq \chi(M) = 2 + c_1(M)^2/2,$$

then $c_1(M)^2 < 0$, and the inequality $\mu_L(M) \geq g - 1$ implies

$$c_1(M) \cdot c_1(N) \geq g + 1 \geq k.$$

From now on, we assume $h^0(S, M) \geq 2$. Since $\omega_C \otimes N^\vee = M \otimes \mathcal{O}_C$, the line bundle $N \otimes \mathcal{O}_C$ contributes to $\text{Cliff}(C)$. The short exact sequence

$$0 \rightarrow M^\vee \rightarrow N \rightarrow N \otimes \mathcal{O}_C \rightarrow 0$$

gives $h^0(C, N \otimes \mathcal{O}_C) \geq h^0(S, N)$. It follows that

$$\begin{aligned} \text{Cliff}(N \otimes \mathcal{O}_C) &= c_1(N) \cdot (c_1(N) + c_1(M)) - 2h^0(C, N \otimes \mathcal{O}_C) + 2 \\ &\leq c_1(N)^2 + c_1(N) \cdot c_1(M) - 2\chi(N) - 2h^1(S, N) + 2 \\ &= -2 + c_1(N) \cdot c_1(M) - 2h^1(S, N). \end{aligned}$$

Since $\text{Cliff}(N \otimes \mathcal{O}_C) \geq k - 2$, then $c_1(M) \cdot c_1(N) \geq k + 2h^1(S, N) \geq k$. \square

Our goal is to count the number of moduli of μ_L -unstable and properly μ_L -semistable LM bundles of rank 2. Fix a nonnegative integer l and a non-trivial, globally generated line bundle N on S such that, if $M := L \otimes N^\vee$, either $\mu_L(M) = \mu_L(N) = g - 1$ or $\mu_L(M) > g - 1 > \mu_L(N)$. We consider the moduli stack $\mathcal{E}_{N,l}$ parametrizing filtrations $0 \subset M \subset E$ with $[M] \in \mathcal{M}(v(M))(\mathbb{C})$ and $[E/M] \in \mathcal{M}(v(N \otimes I_\xi))(\mathbb{C})$, where $l(\xi) = l$. Note that, as N and M are line bundles, the stack $\mathcal{M}(v(M))$ has a unique \mathbb{C} -point endowed with an automorphism group of dimension 1, while $\mathcal{M}(v(N \otimes I_\xi))$ is corepresented by the Hilbert scheme $S^{[l]}$ parametrizing 0-dimensional subschemes of S of length l . Two filtrations $0 \subset M \subset E$ and $0 \subset M' \subset E'$ are equivalent whenever there exists a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & E \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ M' & \longrightarrow & E', \end{array}$$

where φ_1 and φ_2 are two isomorphisms (cf. [2] for the proof that $\mathcal{E}_{N,l}$ is algebraic). The stack $\mathcal{E}_{N,l}$ can be alternatively described as the moduli stack of extensions of type (3.3).

3.5 Stability of Lazarsfeld-Mukai bundles of rank 2

Let $p : \mathcal{E}_{N,l} \rightarrow \mathcal{M}(v(M)) \times \mathcal{M}(v(N \otimes I_{\xi}))$ be the natural morphism of stacks mapping the short exact sequence (3.3) to $(M, N \otimes I_{\xi})$. The fibre of p over the \mathbb{C} -point $(M, N \otimes I_{\xi})$ of $\mathcal{M}(v(M)) \times \mathcal{M}(v(N \otimes I_{\xi}))$ is the quotient stack

$$[\mathrm{Ext}^1(N \otimes I_{\xi}, M) / \mathrm{Hom}(N \otimes I_{\xi}, M)],$$

where the action of $\mathrm{Hom}(N \otimes I_{\xi}, M)$ over $\mathrm{Ext}^1(N \otimes I_{\xi}, M)$ is the trivial one (cf. [2]); it follows that in general p is not representable.

We define $\tilde{P}_{N,l}$ to be the closure of the image of $\mathcal{E}_{N,l}$ under the natural projection $q : \mathcal{E}_{N,l} \rightarrow \mathcal{M}(v(E))$, which maps the point of $\mathcal{E}_{N,l}$ given by (3.3) to $[E]$. The morphism q is representable (cf. proof of Lemma (4.1) in [2]) and the fibre of q over a \mathbb{C} -point of $\tilde{P}_{N,l}$ corresponding to E is the Quot-scheme $\mathrm{Quot}_S(E, P)$, where P is the Hilbert polynomial of $N \otimes I_{\xi}$. We denote by $P_{N,l}$ the open substack of $\tilde{P}_{N,l}$ whose \mathbb{C} -points correspond to vector bundles E satisfying $h^1(S, E) = h^2(S, E) = 0$.

Let $\mathcal{G}_{N,l} \rightarrow P_{N,l}$ be the Grassmann bundle with fibre over a point $[E] \in P_{N,l}(\mathbb{C})$ equal to $G(2, H^0(S, E))$. A \mathbb{C} -point of $\mathcal{G}_{N,l}$ is a pair (E, Λ) and comes endowed with an automorphism group equal to $\mathrm{Aut}(E)$. We consider the rational map

$$h_{N,l} : \mathcal{G}_{N,l} \dashrightarrow \mathcal{W}_d^1(|L|),$$

mapping a general point $(E, \Lambda) \in \mathcal{G}_{N,l}(\mathbb{C})$ to the pair $(C_{\Lambda}, A_{\Lambda})$, where C_{Λ} is the degeneracy locus of the evaluation map $ev_{\Lambda} : \Lambda \otimes \mathcal{O}_S \rightarrow E$, which is injective for a general $\Lambda \in G(2, H^0(S, E))$, and $\omega_{C_{\Lambda}} \otimes A_{\Lambda}^{\vee}$ is the cokernel of ev_{Λ} .

Notice that $d := c_1(N) \cdot c_1(M) + l$. Since while mapping to $\mathcal{W}_d^1(|L|)$ we forget the automorphisms, the fibre of $h_{N,l}$ over (C, A) is the quotient stack

$$[\mathbb{P}(\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^{\vee})^{\circ}) / \mathrm{Aut}(E_{C,A})],$$

where $\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^{\vee})^{\circ}$ denotes the open subgroup of $\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^{\vee})$ consisting of those morphisms whose kernel is isomorphic to $\mathcal{O}_S^{\oplus 2}$, and $\mathrm{Aut}(E_{C,A})$ acts on $\mathbb{P}(\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^{\vee})^{\circ})$ by composition. In particular, $h_{N,l}$ is not representable. As remarked in Section 1.4, one has

$$\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^{\vee}) \simeq H^0(S, E_{C,A} \otimes E_{C,A}^{\vee});$$

it is trivial to check that

$$\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^{\vee})^{\circ} \simeq \mathrm{Aut}(E_{C,A}).$$

Therefore, the action of $\mathrm{Aut}(E_{C,A})$ on $\mathbb{P}(\mathrm{Hom}(E_{C,A}, \omega_C \otimes A^{\vee})^{\circ})$ is transitive and the stabilizer of any point is the subgroup generated by $\mathrm{Id}_{E_{C,A}}$; as a consequence, any fibre of $h_{N,l}$ has dimension -1 (cf. [10] for the definition of the dimension of a locally Noetherian algebraic stack). We denote by $\mathcal{W}_{N,l}$ the closure of the image of $h_{N,l}$. The following holds:

Proposition 3.5.2. *Assume that $P_{N,l}$ be non-empty and let \mathcal{W} be an irreducible component of*

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

$\mathcal{W}_{N,l}$. Then

$$\dim \mathcal{W} \leq g + d - k,$$

where k is the gonality of any curve in $|L|_s$.

Proof. Proposition 3.4.1, together with the fact that $h^0(S, I_\xi) = 0$ if $l > 0$, implies that

$$\dim \operatorname{Hom}(M, N \otimes I_\xi) = \begin{cases} 1 & \text{if } M \simeq N, \xi = \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

It follows that the fibres of p have constant dimension equal to $-\chi(M, N \otimes I_\xi)$, unless $M \simeq N$ and $l = 0$, in which case it is $-\chi(M, N \otimes I_\xi) + 1$.

Regarding the fibres of q , it is well known (cf. [12] Proposition 2.2.8) that, given $[\varphi : E \rightarrow N \otimes I_\xi] \in \operatorname{Quot}_S(E, P)$, the following holds:

$$\begin{aligned} \dim \operatorname{Hom}(K, N \otimes I_\xi) - \dim \operatorname{Ext}^1(K, N \otimes I_\xi) &\leq \dim_{[\xi]} \operatorname{Quot}_S(E, P) \\ &\leq \dim \operatorname{Hom}(K, N \otimes I_\xi), \end{aligned} \quad (3.4)$$

where $K = \ker \varphi$; moreover, if $\operatorname{Ext}^1(K, N \otimes I_\xi) = 0$, then $\operatorname{Quot}_S(E, P)$ is smooth in $[\varphi]$ of dimension equal to $\dim \operatorname{Hom}(K, N \otimes I_\xi)$. Since $K \simeq M$, if $M \simeq N$ and $l = 0$, all the fibres of q are smooth of dimension 1; indeed, one has $\operatorname{Ext}^1(N, N) \simeq H^1(S, \mathcal{O}_S) = 0$. Otherwise, the fibres of q are 0-dimensional. As a consequence, if $P_{N,l}$ is non-empty, then:

$$\begin{aligned} \dim \mathcal{G}_{N,l} &= \dim P_{N,l} + 2(g - d + 1) \\ &= \dim \mathcal{M}(v(M)) + \dim \mathcal{M}(v(N \otimes I_\xi)) \\ &\quad + \langle v(M), v(N \otimes I_\xi) \rangle + 2(g - d + 1) \\ &= 2l - 2 + c_1(M) \cdot c_1(N) - \frac{c_1(M)^2}{2} - \frac{c_1(N)^2}{2} - 2 + l + 2(g - d + 1) \\ &= 3l + 2g - 2d - 2 - (g - 1) + 2c_1(M) \cdot c_1(N) \\ &= g + d - 1 - c_1(N) \cdot c_1(M) \\ &\leq g + d - 1 - k, \end{aligned}$$

where we have used that $c_1(M) + c_1(N) = c_1(L)$ and $d = c_1(M) \cdot c_1(N) + l$, and the last inequality follows from Lemma 3.5.1. The statement is a consequence of the fact that the fibres of $h_{N,l}$ are quotient stacks of dimension equal to -1 . \square

We can finally prove the following result:

Theorem 3.5.3. *Assume that general curves in $|L|_s$ have Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$.*

- *If $\rho(g, 1, d) > 0$, any dominating component of $\mathcal{W}_d^1(|L|)$ corresponds to μ_L -stable LM bundles. In particular, if $C \in |L|_s$ is general, the variety $\mathcal{W}_d^1(C)$ is reduced and has the expected dimension $\rho(g, 1, d)$.*

3.6 Lazarsfeld-Mukai bundles of rank 3 which are not μ_L -stable

- If $\rho(g, 1, k) = 0$ and $C \in |L|_s$ is general, then $W_k^1(C)$ has dimension 0.

Proof. When $\rho(g, 1, d) > 0$, we show that no component \mathcal{W} of $\mathcal{W}_d^1(|L|_s)$ corresponding to either μ_L -unstable or properly μ_L -semistable LM bundles dominates $|L|$. Proposition 3.5.2 gives:

$$\dim \mathcal{W} \leq g + d - k \leq g + d - \frac{g+2}{2}.$$

Our claim follows by remarking that any dominating component of $\mathcal{W}_d^1(|L|)$ has dimension at least $g + \rho(g, 1, d)$ and that $\rho(g, 1, d) > d - \frac{g+2}{2}$ whenever $d > \frac{g+2}{2}$.

If $k = \frac{g+2}{2}$, that is, $\rho(g, 1, k) = 0$, our parameter count shows that any dominating component of $\mathcal{W}_k^1(|L|)$ has dimension g ; hence, if $C \in |L|$ is general, $W_k^1(C)$ is 0-dimensional, even though non necessarily reduced. By induction on d , one excludes the existence of components of $\mathcal{W}_d^1(|L|)$ whose general points correspond to linear series which are not base point free. \square

3.6 Lazarsfeld-Mukai bundles of rank 3 which are not μ_L -stable

We fix a LM bundle $E = E_{C,A}$ associated with a complete, base point free g_d^2 on a smooth connected curve $C \in |L|_s$ with $L \in \text{Ample}(S)$. By Proposition 3.3.1, we have

$$v(E) = 3 + c_1(L) + (2 + g - d)\omega,$$

where $g = g(C)$. We assume that E is not μ_L -stable and, if it is also μ_L -unstable, we look at its HN filtration:

$$0 = E_0 \subset E_1 \subset \dots \subset E_s = E.$$

On the other hand, if E is properly μ_L -semistable, we consider its JH filtration:

$$0 = JH_0(E) \subset JH_1(E) \subset \dots \subset JH_s(E) = E.$$

We first consider the cases where either E is properly μ_L -semistable and $JH_1(E)$ has rank 2, or E is μ_L -unstable, $\text{rk } E_1 = 2$ and E_1 is μ_L -stable. Under these hypotheses, E sits in the following short exact sequence:

$$0 \rightarrow M \rightarrow E \rightarrow N \otimes I_{\xi} \rightarrow 0, \quad (3.5)$$

where $M = JH_1(E)$ (resp. $M = E_1$) is a μ_L -stable vector bundle of rank 2, N is a line bundle and I_{ξ} is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$. Moreover,

$$\mu_L(M) \geq \mu_L(E) = \frac{2g-2}{3} \geq \mu_L(N \otimes I_{\xi}) = \mu_L(N), \quad (3.6)$$

with the former inequality being strict whenever the latter one is. We obtain that $c_1(L) = c_1(E) = c_1(M) + c_1(N)$ and $d = c_2(E) = c_1(N) \cdot c_1(M) + l(\xi) + c_2(M)$, where $l(\xi)$ denotes the length of ξ . We prove the following:

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

Lemma 3.6.1. *Assume a general curve $C \in |L|_s$ has Clifford dimension 1 and gonality k . In the above situation, one has $c_1(N) \cdot c_1(M) \geq k$ and*

$$d \geq \frac{3}{4}k + \frac{7}{6} + \frac{g}{3}. \quad (3.7)$$

Proof. As E is globally generated off a finite number of points, the line bundle N is base point free and non-trivial, thus $h^0(S, N) \geq 2$ and $\mu_L(N) > 0$. The inequality $\mu_L(M) > 0$ implies that $h^2(S, M) = 0$ and, since $\mu_L(\det M) = 2\mu_L(M)$, we have that $h^2(S, \det M) = 0$, too. Therefore,

$$h^0(S, \det M) \geq \chi(\det M) = 2 + c_1(M)^2/2$$

and, if $h^0(S, \det M) < 2$, then $c_1(M)^2 \leq -2$ and $c_1(N) \cdot c_1(M) \geq (4g+2)/3 > k$ by the first inequality in (3.6), which gives

$$c_1(M)^2 + c_1(N) \cdot c_1(M) \geq \frac{4g-4}{3}. \quad (3.8)$$

On the other hand, if $h^0(S, \det M) \geq 2$, then $N \otimes \mathcal{O}_C$ contributes to $\text{Cliff}(C)$ and one shows, as in the proof of Lemma 3.5.1, that $c_1(N) \cdot c_1(M) \geq k + 2h^1(S, N) \geq k$.

The μ_L -stability of M implies that

$$-2 \leq \langle v(M), v(M) \rangle = c_1(M)^2 - 4\chi(M) + 8 = 4c_2(M) - c_1(M)^2 - 8.$$

Therefore, we have

$$d = c_1(N) \cdot c_1(M) + c_2(M) + l(\xi) \geq c_1(N) \cdot c_1(M) + \frac{c_1(M)^2}{4} + \frac{6}{4} \geq \frac{3}{4}k + \frac{7}{6} + \frac{g}{3};$$

this concludes the proof. \square

Now, we assume that either E is μ_L -unstable, $\text{rk } E_1 = 1$ and E/E_1 is μ_L -stable, or E is properly μ_L -semistable and its JH filtration is of type $0 \subset JH_1(E) \subset E$ with $JH_1(E)$ of rank 1. Denoting by N the line bundle E_1 (resp. $JH_1(E)$), one has a short exact sequence:

$$0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0, \quad (3.9)$$

where E/N is a rank-2, μ_L -stable, torsion free sheaf on S such that

$$\mu_L(N) \geq \mu_L(E) \geq \mu_L(E/N),$$

and either both inequalities are strict, or none is. We prove the following:

Lemma 3.6.2. *In the above situation, if a general curve $C \in |L|_s$ has Clifford dimension 1 and gonality k , then $c_1(N) \cdot c_1(E/N) \geq k$.*

3.6 Lazarsfeld-Mukai bundles of rank 3 which are not μ_L -stable

Proof. As in the proof of Lemma 3.4.2 one shows that $h^0(S, E/N) \geq 2$. Since E/N is stable, then $\mu_L(E/N) > 0$ and $h^2(S, E/N) = 0$. Moreover, the vector bundle $(E/N)^{\vee\vee}$ is globally generated off a finite number of points and $h^0(S, \det(E/N)) \geq 2$ by Lemma 3.4.3 because $\det(E/N) := \det(E/N)^{\vee\vee}$.

Since $\mu_L(N) = c_1(N) \cdot (c_1(N) + c_1(E/N)) \geq (2g - 2)/3 > 0$, we have $h^2(S, N) = 0$. Hence, if $h^0(S, N) < 2$, then $c_1(N)^2 < 0$ and $c_1(N) \cdot c_1(E/N) \geq (2g + 4)/3 > k$. Otherwise, the restriction $N \otimes \mathcal{O}_C$ contributes to the Clifford index and this implies $c_1(N) \cdot c_1(E/N) \geq k$, too. \square

The cases still to be considered are the following ones:

- (i) E is μ_L -unstable with HN filtration $0 \subset E_1 \subset E_2 \subset E$.
- (ii) E is properly μ_L -semistable with JH filtration $0 \subset JH_1(E) \subset JH_2(E) \subset E$.
- (iii) E is μ_L -unstable with HN filtration $0 \subset E_1 \subset E$ and E_1 is a properly μ_L -semistable vector bundle of rank 2.
- (iv) E is μ_L -unstable with HN filtration $0 \subset E_1 \subset E$ and E_1 is a line bundle such that E/E_1 is a properly μ_L -semistable torsion free sheaf of rank 2.

In all these cases one has four short exact sequences:

$$0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0 \quad (3.10)$$

$$0 \rightarrow M \rightarrow E \rightarrow N_1 \otimes I_{\zeta_1} \rightarrow 0, \quad (3.11)$$

$$0 \rightarrow N \rightarrow M \rightarrow N_2 \otimes I_{\zeta_2} \rightarrow 0, \quad (3.12)$$

$$0 \rightarrow N_2 \otimes I_{\zeta_2} \rightarrow E/N \rightarrow N_1 \otimes I_{\zeta_1} \rightarrow 0, \quad (3.13)$$

where N, N_1, N_2 are line bundles, I_{ζ_1} and I_{ζ_2} denote the ideal sheaves of two zero-dimensional subschemes $\zeta_1, \zeta_2 \subset S$, the sheaf E/N has rank-2 and no torsion, while M is a vector bundle of rank 2. Moreover, the following inequalities hold:

$$\mu_L(N) \geq \mu_L(N_2) \geq \mu_L(N_1), \quad (3.14)$$

$$\mu_L(N) \geq \frac{2g-2}{3} \geq \mu_L(N_1); \quad (3.15)$$

in particular, $\mu_L(N) = \mu_L(N_2)$ (resp. $\mu_L(N_1) = \mu_L(N_2)$) whenever M (resp. E/N) is properly μ_L -semistable, that is, in cases (ii) and (iii) (resp. in cases (ii) and (iv)). Analogously, equalities in (3.15) force E to be properly μ_L -semistable with JH-filtration $0 \subset N \subset M \subset E$, that is, one is in case (ii).

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

Lemma 3.6.3. *In the above situation, $N_1 \otimes \mathcal{O}_C$ always contributes to the Clifford index of $C \in |L|_s$. Moreover, one of the following occurs:*

- (a) *Both $N \otimes \mathcal{O}_C$ and $N_1 \otimes \mathcal{O}_C$ contribute to the Clifford index of $C \in |L|_s$.*
- (b) *The inequality $c_1(N) \cdot (c_1(N_1) + c_1(N_2)) \geq \frac{2g+4}{3}$ holds and either $N_2 \otimes \mathcal{O}_C$ contributes to the Clifford index of C or $c_1(N_2) \cdot (c_1(N) + c_1(N_1)) \geq g$.*
- (c) *The linear series $N \otimes \mathcal{O}_C$ contributes to the Clifford index of $C \in |L|_s$ and one has the inequality $c_1(N_2) \cdot c_1(N) > \frac{1}{2}c_1(N) \cdot (c_1(N_1) + c_1(N_2))$.*
- (d) *The inequality $c_1(N) \cdot c_1(N_2) \geq \frac{g+5}{3}$ holds.*

In particular, if a general $C \in |L|_s$ has Clifford dimension 1 and gonality k , then

$$d \geq c_1(N) \cdot c_1(N_1) + c_1(N) \cdot c_1(N_2) + c_1(N_1) \cdot c_1(N_2) \geq \frac{3}{2}k. \quad (3.16)$$

Proof. Being a quotient of E off a finite set, N_1 is base point free and non-trivial, thus $h^0(S, N_1) \geq 2$ and $\mu_L(N_1) > 0$. By the Strong Bertini's Theorem (cf. [24]), N_1 is nef. Proposition 3.4.1 implies $h^2(S, N) = h^2(S, N_2) = 0$ because of (3.14). Analogously, $\mu_L(N_2 \otimes N) = \mu_L(N_2) + \mu_L(N) > 0$ and $h^2(S, N_2 \otimes N) = 0$. Moreover, the following holds:

$$\begin{aligned} c_1(N_2 \otimes N)^2 &= c_1(N_2)^2 + c_1(N)^2 + 2c_1(N_2) \cdot c_1(N) \\ &\geq c_1(N)^2 + c_1(N_2) \cdot c_1(N) + c_1(N_1) \cdot c_1(N) + c_1(N_1)^2 \\ &= \mu_L(N) + c_1(N_1)^2 > 0, \end{aligned}$$

where we have used that, since $\mu_L(N_2) \geq \mu_L(N_1)$, then

$$c_1(N_2)^2 + c_1(N_2) \cdot c_1(N) \geq c_1(N_1)^2 + c_1(N_1) \cdot c_1(N), \quad (3.17)$$

and that $c_1(N_1)^2 \geq 0$ because N_1 is nef. We obtain that

$$h^0(S, N_2 \otimes N) \geq \chi(N_2 \otimes N) = 2 + \frac{1}{2}c_1(N_2 \otimes N)^2 > 2,$$

thus $N_1 \otimes \mathcal{O}_C$ always contributes to the Clifford index of $C \in |L|_s$.

If both $h^0(S, N_2) \geq 2$ and $h^0(S, N) \geq 2$, we are in case (a).

If $h^0(S, N_2) \geq 2$ and $h^0(S, N) < 2$, we show that (b) occurs. Since $\chi(N) < 2$, one has $c_1(N)^2 < 0$ and $c_1(N) \cdot (c_1(N_1) + c_1(N_2)) \geq \mu_L(E) + 2 = (2g + 4)/3$ by the first inequality in (3.15). Since $\mu_L(N \otimes N_1) > 0$, then $h^2(S, N \otimes N_1) = 0$. Moreover, one can show that

$$c_1(N \otimes N_1)^2 \geq \mu_L(N_1) + c_1(N_2)^2 > c_1(N_2)^2.$$

It follows that, if $c_1(N \otimes N_1)^2 < 0$, then $c_1(N_2)^2 < 0$ and

$$2g - 2 < 2c_1(N) \cdot c_1(N_2) + 2c_1(N_1) \cdot c_1(N_2),$$

3.6 Lazarsfeld-Mukai bundles of rank 3 which are not μ_L -stable

that is, $c_1(N_2) \cdot (c_1(N) + c_1(N_1)) \geq g$. On the other hand, if $c_1(N \otimes N_1)^2 \geq 0$, then $h^0(S, N \otimes N_1) \geq 2$ and $N_2 \otimes \mathcal{O}_C$ contributes to the Clifford index.

From now on, let $h^0(S, N_2) < 2$, hence $c_1(N_2)^2 < 0$. Since $\det E/N \simeq N_1 \otimes N_2$, Lemma 3.4.3 implies $h^0(S, N_1 \otimes N_2) \geq 2$. Thus, if $h^0(S, N) \geq 2$, the linear series $N \otimes \mathcal{O}_C$ contributes to the Clifford index of $C \in |L|_s$. Furthermore, inequality (3.17), together with the fact that $c_1(N_2)^2 < 0 \leq c_1(N_1)^2$, implies that

$$c_1(N_2) \cdot c_1(N) > \frac{1}{2}c_1(N) \cdot (c_1(N_1) + c_1(N_2)),$$

and we are in case (c).

It remains to treat the case where both $h^0(S, N_2) < 2$ and $h^0(S, N) < 2$. Under these hypotheses, $c_1(N_2)^2 < 0$ and $c_1(N)^2 < 0$ and we obtain

$$\begin{aligned} 2g - 2 &\leq c_1(N_1)^2 + 2c_1(N_1) \cdot c_1(N) + 2c_1(N_1) \cdot c_1(N_2) + 2c_1(N) \cdot c_1(N_2) - 4 \\ &= 2c_1(N) \cdot c_1(N_2) + 2\mu_L(N_1) - c_1(N_1)^2 - 4 \\ &\leq 2c_1(N) \cdot c_1(N_2) + \frac{4g - 4}{3} - 4. \end{aligned}$$

As a consequence, $c_1(N) \cdot c_1(N_2) \geq \frac{g+5}{3}$ and we are in case (d).

Now, we assume that C has Clifford dimension 1 and gonality k and prove inequality (3.16). One shows, as in Lemma 3.5.1, that

$$c_1(N_1) \cdot (c_1(N) + c_1(N_2)) \geq k, \quad (3.18)$$

because $N_1 \otimes \mathcal{O}_C$ always contributes to the Clifford index of $C \in |L|_s$. Analogously, if $N \otimes \mathcal{O}_C$ (resp. $N_2 \otimes \mathcal{O}_C$) contributes to $\text{Cliff}(C)$, then

$$c_1(N) \cdot (c_1(N_1) + c_1(N_2)) \geq k \quad (\text{resp. } c_1(N_2) \cdot (c_1(N) + c_1(N_1)) \geq k);$$

therefore, the last part of the statement is proved if either (a) or (b) occurs (use that $(2g + 4)/3 \geq k$).

In case (c), one arrives at the same conclusion by adding inequality (3.18) and

$$c_1(N) \cdot c_1(N_2) > \frac{1}{2}c_1(N) \cdot (c_1(N_1) + c_1(N_2)) \geq \frac{k}{2}. \quad (3.19)$$

Similarly, in case (d), one uses that $c_1(N) \cdot c_1(N_2) \geq (g + 5)/3 \geq k/2$.

□

Corollary 3.6.4. *Assume $C \in |L|_s$ has Clifford dimension 1 and maximal gonality $k = \lfloor \frac{g+3}{2} \rfloor$ and let E be the Lazarsfeld-Mukai bundle associated with a complete, base point free net $A \in W_d^2(C)$.*

If E is not μ_L -stable, $d < \frac{3}{4}k + \frac{7}{6} + \frac{8}{3}$ and $(g, d) \neq (6, 6)$, then E is given by an extension of type (3.9), with $N \in \text{Pic}(S)$ and E/N a μ_L -stable, torsion free sheaf of rank 2 such that $\mu_L(N) \geq (2g - 2)/3 \geq \mu_L(E/N)$.

Proof. Apply Lemma 3.6.1 and Lemma 3.6.3 and use that $\lceil \frac{3}{4}k + \frac{7}{6} + \frac{g}{3} \rceil \leq \lceil \frac{3}{2}k \rceil$ unless $g = 6$. \square

3.7 Cases with a μ_L -stable subbundle of rank 2 and L -slope $\geq \mu_L(E)$

We assume that a general curve in $|L|$ has Clifford dimension 1 and maximal gonality. In this section we show that, if $C \in |L|_s$ is general, the LM bundle E corresponding to a general, complete, base point free g_d^2 on C is neither properly μ_L -semistable with JH filtration $0 \subset JH_1(E) \subset E$ and $\text{rk } JH_1(E) = 2$, nor μ_L -unstable with a μ_L -stable, rank-2 vector bundle E_1 as maximal destabilizing sheaf.

Fix a positive integer d . Choose $l \in \mathbb{N}$ and a non-trivial, globally generated line bundle N such that

$$\mu_L(N) \leq \frac{2g-2}{3} \leq \frac{(c_1(L) - c_1(N)) \cdot c_1(L)}{2}, \quad (3.20)$$

and impose that these are either two equalities or two strict inequalities. Set

$$\begin{aligned} c_1 &:= c_1(L) - c_1(N), \\ c_2 &:= d - c_1 \cdot c_1(N) - l, \\ \chi &:= g - d + 5 - \chi(N) + l, \end{aligned}$$

and define the vector $v := 2 + c_1 + (\chi - 2)\omega \in H^*(S, \mathbb{Z})$. The following construction is analogous to that of Section 3.5.

Let $\mathcal{E}_{N,l}$ be the moduli stack of filtrations $0 \subset M \subset E$, where $[M] \in \mathcal{M}_L(v)^{\mu s}(\mathbb{C})$ and $[E/M] \in \mathcal{M}(v(N \otimes I_\xi))(\mathbb{C})$ with $l(\xi) = l$. This is alternatively described as the moduli stack of extensions

$$0 \rightarrow M \rightarrow E \rightarrow N \otimes I_\xi \rightarrow 0, \quad (3.21)$$

with M and ξ as above.

If $p : \mathcal{E}_{N,l} \rightarrow \mathcal{M}_L(v)^{\mu s} \times \mathcal{M}(v(N \otimes I_\xi))$ denotes the morphism of Artin stacks mapping the short exact sequence (3.21) to $(M, N \otimes I_\xi)$, the fibre of p over the point of $\mathcal{M}_L(v)^{\mu s} \times \mathcal{M}(v(N \otimes I_\xi))$ corresponding to the pair $(M, N \otimes I_\xi)$ is the quotient stack

$$[\text{Ext}^1(N \otimes I_\xi, M) / \text{Hom}(N \otimes I_\xi, M)].$$

Define $\tilde{P}_{N,l}$ to be the closure of the image of $\mathcal{E}_{N,l}$ under the natural projection

$$q : \mathcal{E}_{N,l} \rightarrow \mathcal{M}(v(E)),$$

which sends the isomorphism class of extension (3.21) to $[E] \in \mathcal{M}(v(E))(\mathbb{C})$. The morphism q is representable and the fibre of q over the point of $\tilde{P}_{N,l}$ corresponding to $[E]$ is the Quot-scheme $\text{Quot}_S(E, P)$, where by P we denote the Hilbert polynomial of $N \otimes I_\xi$. We consider the open substack $P_{N,l} \subset \tilde{P}_{N,l}$, whose \mathbb{C} -points are isomorphism classes of

3.7 Cases with a μ_L -stable subbundle of rank 2 and L -slope $\geq \mu_L(E)$

vector bundles E such that $h^1(S, E) = h^2(S, E) = 0$.

Lemma 3.7.1. *The stack $P_{N,l}$, if nonempty, has dimension*

$$\dim P_{N,l} = 2l + \langle v, v \rangle + \langle v(N \otimes I_{\xi}), v \rangle.$$

Proof. We claim that the dimension of the fibres of p is constant. Indeed, Serre duality and Proposition 3.4.1 imply that for any $[M] \in \mathcal{M}_L(v)^{\mu_S}(\mathbb{C})$ and $\xi \in S^{[l]}$ one has $\dim \operatorname{Ext}^2(N \otimes I_{\xi}, M) = \dim \operatorname{Hom}(M, N \otimes I_{\xi}) = 0$. This shows that $\mathcal{E}_{N,l}$, if nonempty, has dimension equal to

$$\dim(M_L(v)^{\mu_S} \times \mathcal{M}(v(N \otimes I_{\xi}))) - \chi(N \otimes I_{\xi}, M) = 2l - 1 + 1 + \langle v, v \rangle + \langle v(N \otimes I_{\xi}), v \rangle;$$

note that this coincides with the dimension computed by Yoshioka (cf. Lemma 5.2 in [26]). The statement follows now by remarking that, when $P_{N,l}$ is nonempty, we have $\dim P_{N,l} = \dim \tilde{P}_{N,l} = \dim \mathcal{E}_{N,l}$ because the Quot-schemes corresponding to the fibres of q are 0-dimensional (use inequalities analogous to (3.4)). \square

We consider the Grassmann bundle $\mathcal{G}_{N,l} \rightarrow P_{N,l}$, whose fibre over $[E] \in P_{N,l}(\mathbb{C})$ is $G(3, H^0(S, E))$, and the rational map $h_{N,l} : \mathcal{G}_{N,l} \dashrightarrow \mathcal{W}_d^2(|L|)$. The fibre of $h_{N,l}$ over a pair (C, A) is the quotient stack

$$[\mathbb{P}(\operatorname{Hom}(E_{C,A}, \omega_C \otimes A^{\vee})^{\circ}) / \operatorname{Aut}(E_{C,A})],$$

where $\operatorname{Hom}(E_{C,A}, \omega_C \otimes A^{\vee})^{\circ} \subset \operatorname{Hom}(E_{C,A}, \omega_C \otimes A^{\vee})$ consists, by definition, of morphisms with kernel isomorphic to $\mathcal{O}_S^{\oplus 3}$. Such quotient stack has dimension equal to -1 , as in Section 3.5. Our goal is to estimate the dimension of the closure of the image of $h_{N,l}$, which is denoted by $\mathcal{W}_{N,l}$. We first prove the following:

Lemma 3.7.2. *If $\mathcal{G}_{N,l}$ is nonempty, then*

$$\dim \mathcal{G}_{N,l} = g + \rho(g, 2, d) + \chi(M, N \otimes I_{\xi}).$$

Moreover, $\chi(M, N \otimes I_{\xi}) \leq \frac{4}{3}g + \frac{8}{3} - d - \frac{3}{2}c_1(N) \cdot c_1$.

Proof. We use that

$$\begin{aligned} 2(\rho(g, 2, d) - 1) &= \langle v(E), v(E) \rangle \\ &= \langle v(N \otimes I_{\xi}), v(N \otimes I_{\xi}) \rangle + \langle v, v \rangle + 2\langle v(N \otimes I_{\xi}), v \rangle \\ &= 2l - 2 + \langle v, v \rangle + 2\langle v(N \otimes I_{\xi}), v \rangle; \end{aligned}$$

this implies that

$$\begin{aligned} \dim \mathcal{G}_{N,l} &= \dim P_{N,l} + 3(h^0(S, E) - 3) \\ &= 2\rho(g, 2, d) - \langle v(N \otimes I_{\xi}), v \rangle + 3(g - d + 2) \\ &= g + \rho(g, 2, d) + \chi(M, N \otimes I_{\xi}), \end{aligned}$$

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

as soon as $\mathcal{G}_{N,l}$ is nonempty.

Since $\chi(M, N \otimes I_{\xi}) = -\langle v(N \otimes I_{\xi}), v \rangle = 2\chi(N \otimes I_{\xi}) + \chi - 4 - c_1(N) \cdot c_1$, the last part of the statement follows by remembering that $\chi(E) = \chi + \chi(N \otimes I_{\xi}) = g - d + 5$ and that

$$\frac{c_1(N)^2}{2} \leq \frac{g-1}{3} - \frac{c_1(N) \cdot c_1}{2}$$

because $\mu_L(E) \geq \mu_L(N \otimes I_{\xi})$. \square

In conclusion, we prove the following:

Proposition 3.7.3. *Assume that a general curve in $|L|_s$ has Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$. Let $\mathcal{W} \subset \mathcal{W}_{N,l}$ be an irreducible component of $\mathcal{W}_d^2(|L|)$; then, $\rho(g, 2, d) > 0$ and \mathcal{W} does not dominate the linear system $|L|$.*

Proof. Lemma 3.6.1 gives $c_1(N) \cdot c_1 \geq k \geq (g+2)/2$ and $d \geq \frac{3}{4}k + \frac{7}{6} + \frac{g}{3} \geq \frac{17}{24}g + \frac{23}{12}$; in particular, $\rho(g, 2, d) \geq 0$. By Lemma 3.7.2, we have

$$\begin{aligned} \dim \mathcal{G}_{N,l} &\leq g + \rho(g, 2, d) + \frac{4}{3}g + \frac{8}{3} - d - \frac{3}{2}k \\ &\leq g + \rho(g, 2, d) + \frac{4}{3}g + \frac{8}{3} - d - \frac{3}{4}g - \frac{3}{2} \\ &= g + \rho(g, 2, d) + \frac{7}{12}g + \frac{7}{6} - d. \end{aligned}$$

Since any fibre of $h_{N,l}$ is an algebraic stack of dimension -1 , then

$$\dim \mathcal{W} \leq g + \rho(g, 2, d) + \frac{7}{12}g + \frac{13}{6} - d.$$

The right hand side is strictly smaller than $g + \rho(g, 2, d)$ because $d > \frac{7g+26}{12}$. It follows that \mathcal{W} cannot dominate $|L|$. \square

3.8 Cases with a μ_L -stable quotient sheaf of rank 2 and L -slope $\leq \mu_L(E)$

In this section we count the number of moduli of rank-3 LM bundles E , which are either properly μ_L -semistable with JH filtration $0 \subset JH_1(E) \subset E$ where $JH_1(E)$ is a line bundle, or μ_L -unstable with maximal destabilizing sheaf E_1 such that E/E_1 is a μ_L -stable, torsion free sheaf of rank 2.

Fix an integer $d \geq 4$. Choose $N \in \text{Pic}(S)$ such that

$$\mu_L(N) \geq \frac{2g-2}{3} \geq \frac{(c_1(L) - c_1(N)) \cdot c_1(L)}{2}, \quad (3.22)$$

with equality holding either everywhere or nowhere.

3.8 Cases with a μ_L -stable quotient sheaf of rank 2 and L -slope $\leq \mu_L(E)$

As before, we set $c'_1 := c_1(L) - c_1(N)$, $c'_2 := d - c'_1 \cdot c_1(N)$, $\chi' := g - d + 5 - \chi(N)$, $v' := 2 + c'_1 + (\chi' - 2)\omega \in H^*(S, \mathbb{Z})$.

We denote by \mathcal{F}_N the algebraic stack of extensions

$$0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0, \quad (3.23)$$

where E/N defines a point of $\mathcal{M}_L^{\mu_S}(v')$. Equivalently, \mathcal{F}_N is the moduli stack of filtrations $0 \subset N \subset E$ such that $[E/N] \in \mathcal{M}_L^{\mu_S}(v')(\mathbb{C})$. Consider the two projections $p : \mathcal{F}_N \rightarrow \mathcal{M}_L^{\mu_S}(v') \times \mathcal{M}(v(N))$ and $q : \mathcal{F}_N \rightarrow \mathcal{M}(v(E))$ and define \tilde{R}_N to be the closure of the image of q . The open substack $R_N \subset \tilde{R}_N$ consists, by definition, of points corresponding to bundles E such that $h^i(S, E) = 0$, $i = 1, 2$. We look at the Grassmann bundle $\mathcal{G}_N \rightarrow R_N$ with fibre over $[E] \in R_N(\mathbb{C})$ equal to $G(3, H^0(S, E))$. The closure of the image of \mathcal{G}_N under the rational map $h_N : \mathcal{G}_N \dashrightarrow \mathcal{W}_d^2(|L|)$ is denoted by \mathcal{W}_N . As before, the fibres of h_N are quotient stacks of dimension -1 .

Lemma 3.8.1. *The stack \mathcal{G}_N , if nonempty, has dimension*

$$\dim \mathcal{G}_N = g + \rho(g, 2, d) + \chi(E/N, N).$$

Proof. The fibre of p over a point of $\mathcal{M}_L^{\mu_S}(v') \times \mathcal{M}(v(N))$ corresponding to $(E/N, N)$ is the quotient stack $[\text{Ext}^1(E/N, N)/\text{Hom}(E/N, N)]$.

Since E/N is μ_L -stable and $\mu_L(N) \geq \mu_L(E/N)$, Serre duality and Proposition 3.4.1 imply that $\text{Ext}^2(E/N, N) = 0$; hence, the dimension of the fibres of p is constantly equal to $-\chi(E/N, N) = \langle v(N), v' \rangle$. The morphism q is representable and, as in the previous sections, one shows that its fibres are Quot-schemes of dimension 0. Therefore, if R_N is nonempty, one has:

$$\dim R_N = \dim \tilde{R}_N = \dim \mathcal{F}_N = \langle v', v' \rangle + \langle v(N), v' \rangle.$$

The statement follows by proceeding as in the proof of Lemma 3.7.2. \square

Next Lemma gives an upper bound for $\chi(E/N, N)$.

Lemma 3.8.2. *Assume that general curves in $|L|$ have Clifford dimension 1 and maximal gonality $k = \lfloor \frac{g+3}{2} \rfloor$. If R_N is nonempty, then $\chi(E/N, N) \leq \frac{3}{2}g - 2d + 3$ for any E/N corresponding to a point of $\mathcal{M}_L^{\mu_S}(v')$.*

Proof. Consider the extension (3.23), where $[E] \in R_N(\mathbb{C})$. Since $\mu_L(N) > 0$, one has $h^1(S, E/N) = h^2(S, N) = 0$. By Lemma 3.4.2, one finds $\chi(E/N) = h^0(S, E/N) \geq 2$, hence $\chi(N) = \chi(E) - \chi(E/N) \leq g - d + 3$. As a consequence:

$$\begin{aligned} \chi(E/N, N) &= 2\chi(N) + \chi' - 4 - c_1(N) \cdot c'_1 \\ &= g - d + 1 + \chi(N) - c_1(N) \cdot c'_1 \\ &\leq 2g - 2d + 4 - c_1(N) \cdot c'_1 \\ &\leq \frac{3}{2}g - 2d + 3, \end{aligned}$$

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

where the last inequality follows from Lemma 3.6.2. \square

Finally, we prove the following:

Proposition 3.8.3. *We assume that a general curve in $|L|$ has Clifford dimension 1 and maximal gonality $k = \lfloor \frac{g+3}{2} \rfloor$. If $d > \frac{3}{4}g + 2$, no irreducible component \mathcal{W} of $\mathcal{W}_d^2(|L|)$, which is contained in \mathcal{W}_N , dominates the linear system $|L|$.*

Proof. Let $\mathcal{W} \subset \mathcal{W}_N$ be an irreducible component of $\mathcal{W}_d^2(|L|)$. Since any fibre of h_N is an Artin stack of dimension equal to -1 , Lemma 3.8.1 and Lemma 3.8.2 imply that

$$\dim \mathcal{W} \leq g + \rho(g, 2, d) + \frac{3}{2}g - 2d + 4.$$

If $\rho(g, 2, d) \geq 0$, the condition $d > \frac{3}{4}g + 2$ prevents the map $\mathcal{W} \rightarrow |L|$ from being dominant. \square

Now we show that, if d is small enough and $C \in |L|_s$, any complete base point free g_d^2 on C , whose LM bundle is given by an extension of type (3.23), is contained in a linear series which is induced from a line bundle on S .

Proposition 3.8.4. *Let L be as in Proposition 3.8.3 and A be a complete, base point free g_d^2 on a curve $C \in |L|_s$. If $d < (5g + 13)/6$ and the LM bundle $[E_{C,A}] \in R_N(\mathbb{C})$ for some $N \in \text{Pic}(S)$, the linear system $|A|$ is contained in the restriction to C of the linear system $|L \otimes N^\vee|$ on S . Moreover, $L \otimes N^\vee$ is adapted to $|L|$ and $\text{Cliff}(L \otimes N^\vee \otimes \mathcal{O}_C) \leq \text{Cliff}(A) = d - 4$.*

Proof. By hypothesis, $E = E_{C,A}$ sits in a short exact sequence like (3.23), where E/N is μ_L -stable and $\mu_L(N) \geq (2g - 2)/3 \geq \mu_L(E/N)$. Since $\mu_L(N) > 0$, then $h^2(S, N) = 0$.

The μ_L -stability of E/N implies

$$-2 \leq \langle v', v' \rangle = 4c'_2 - (c'_1)^2 - 8,$$

thus $c'_2 \geq 3/2 + (c'_1)^2/4$.

If $h^0(S, N) < 2$, then $c_1(N)^2 \leq -2$, which implies $(c'_1)^2 + 2c_1(N) \cdot c'_1 \geq 2g$ and $c'_1 \cdot c_1(N) \geq (2g + 4)/3$. In particular,

$$d = c'_1 \cdot c_1(N) + c'_2 \geq c'_1 \cdot c_1(N) + \frac{3}{2} + \frac{(c'_1)^2}{4} \geq \frac{g}{2} + \frac{3}{2} + \frac{g+2}{3} \geq \frac{5g+13}{6},$$

hence a contradiction. Therefore, both $h^0(S, N) \geq 2$ and $h^0(S, \det E/N) \geq 2$.

Observe that $(E/N)^{\vee\vee}$ is globally generated off a finite set and

$$h^i(S, (E/N)^{\vee\vee}) = h^i(S, E/N) = 0 \text{ for } i = 1, 2.$$

Since the line bundle $\det E/N = \det(E/N)^{\vee\vee}$ is base point free and non trivial, if $h^1(S, \det E/N) \neq 0$, then $(c'_1)^2 = 0$ and Proposition (1.1) in [11] implies the existence of a smooth elliptic curve $\Sigma \subset S$ such that

$$(E/N)^{\vee\vee} = \mathcal{O}_S(\Sigma) \oplus \mathcal{O}_S(\Sigma).$$

Such equality would contradict the stability of E/N , thus we conclude that $(c'_1)^2 \geq 2$ (and $c'_2 \geq 2$) and

$$h^1(S, \det E/N) = 0. \quad (3.24)$$

This ensures that $h^0(C, \det E/N \otimes \mathcal{O}_C)$ does not depend on the curve $C \in |L|_s$ (cf. [5] Lemma (5.2)). Hence, the line bundle $\det E/N = L \otimes N^\vee$ is adapted to $|L|$. We obtain:

$$\begin{aligned} \text{Cliff}(\det E/N \otimes \mathcal{O}_C) &= c_1(E/N)^2 + c_1(N) \cdot c_1(E/N) - 2h^0(C, \det E/N \otimes \mathcal{O}_C) + 2 \\ &\leq c_1(E/N)^2 + c_1(N) \cdot c_1(E/N) - 2h^0(S, \det E/N) + 2 \\ &= c_1(N) \cdot c_1(E/N) - 2 - 2h^1(S, \det E/N) \\ &= d - c_2(E/N) - 2 \\ &\leq d - 4. \end{aligned}$$

It remains only to prove that $h^0(C, \det E/N \otimes \mathcal{O}_C \otimes A^\vee) > 0$. Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(C, A)^\vee \otimes \mathcal{O}_S & \longrightarrow & E & \xrightarrow{\alpha} & \omega_C \otimes A^\vee \longrightarrow 0 \\ & & & & \uparrow \gamma & & \\ & & & & N & & \end{array}$$

Since $h^2(S, N) = 0$, the composition $\alpha \circ \gamma$ is non-zero. Thus, $\text{Hom}(N, \omega_C \otimes A^\vee) \neq 0$ and we have finished because $N^\vee \otimes \omega_C \otimes A^\vee \simeq \det E/N \otimes \mathcal{O}_C \otimes A^\vee$. \square

3.9 Remaining cases

In this section we consider rank-3 LM bundles E of type (i), (ii), (iii), (iv) on page 63, such that $\det E = L$ and $c_2(E) = d$ is fixed.

Choose $l_2 \in \mathbb{N}$ and two line bundles $N, N_2 \in \text{Pic}(S)$ such that $N_1 := L \otimes (N \otimes N_2)^\vee$ is globally generated and non-trivial, and the following holds:

$$\mu_L(N) \geq \mu_L(N_2) \geq \mu_L(N_1), \quad (3.25)$$

$$\mu_L(N) \geq \frac{2g-2}{3} \geq \mu_L(N_1), \quad (3.26)$$

where in (3.26) either both the inequalities are strict, or none is.

Set $v := v(N)$, $v_1 := v(N_1 \otimes I_{\xi_1})$ and $v_2 := v(N_2 \otimes I_{\xi_2})$, with $l(\xi_2) = l_2$ and

$$l(\xi_1) = l_1 := d - l_2 - c_1(N) \cdot c_1(N_1) - c_1(N) \cdot c_1(N_2) - c_1(N_1) \cdot c_1(N_2).$$

Define $\mathcal{F}_{N, N_2, l_2}$ to be the moduli stack of extensions

$$0 \rightarrow N_2 \otimes I_{\xi_2} \rightarrow E/N \rightarrow N_1 \otimes I_{\xi_1} \rightarrow 0,$$

where $\xi_i \subset S$ is a 0-dimensional subscheme of length l_i for $i = 1, 2$. We consider the

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

projections $p_2 : \mathcal{F}_{N,N_2,l_2} \rightarrow \mathcal{M}(v_2) \times \mathcal{M}(v_1)$ and $q_2 : \mathcal{F}_{N,N_2,l_2} \rightarrow \mathcal{M}(v(E/N))$, and we denote by Q_{N,N_2,l_2} the closure of the image of q_2 .

Let \mathcal{E}_{N,N_2,l_2} be the moduli stack of extensions

$$0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0, \text{ with } [E/N] \in Q_{N,N_2,l_2}(\mathbb{C}),$$

and $p_1 : \mathcal{E}_{N,N_2,l_2} \rightarrow \mathcal{M}(v) \times Q_{N,N_2,l_2}$ and $q_1 : \mathcal{E}_{N,N_2,l_2} \rightarrow \mathcal{M}(v(E))$ be the two projections. The closure of the image of q_1 is denoted by \tilde{P}_{N,N_2,l_2} and its open substack, consisting of points which correspond to vector bundles E such that $h^1(S, E) = h^2(S, E) = 0$, by P_{N,N_2,l_2} .

Notice that, if E is a LM bundle of type (i), (ii), (iii), (iv), there exist N, N_2 and l_2 such that $[E]$ defines a point of P_{N,N_2,l_2} . In order to count the number of moduli of such bundles, we start by proving the following:

Lemma 3.9.1. *The stack Q_{N,N_2,l_2} , if nonempty, has dimension*

$$\dim Q_{N,N_2,l_2} = 2l_1 + 2l_2 - 2 + \langle v_1, v_2 \rangle,$$

unless $N_1 \simeq N_2$, $l_2 \neq 0$ and $l_1 = 0$. In this case, for any component $Q \subset Q_{N,N_2,l_2}$, the following inequality holds:

$$\dim Q \leq 2l_1 + 2l_2 - 1 + \langle v_1, v_2 \rangle.$$

Proof. The fibre of p_2 over the point of $\mathcal{M}_L(v_2) \times \mathcal{M}_L(v_1)$ corresponding to the pair $(N_2 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1})$ is the quotient stack

$$[\text{Ext}^1(N_1 \otimes I_{\xi_1}, N_2 \otimes I_{\xi_2}) / \text{Hom}(N_1 \otimes I_{\xi_1}, N_2 \otimes I_{\xi_2})].$$

Since $\mu_L(N_2) \geq \mu_L(N_1)$, if either $N_1 \not\simeq N_2$ or $N_1 \simeq N_2$, $l_1 \neq 0$ and $l_2 = 0$, one finds that

$$\text{Hom}(N_2 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1}) = 0.$$

In these cases, the dimension of any fibre of p_2 equals $-\chi(N_1 \otimes I_{\xi_1}, N_2 \otimes I_{\xi_2})$, while the fibres of q_2 are 0-dimensional Quot-schemes, hence the statement follows.

If $N_1 \simeq N_2$ and $l_1 = l_2 = 0$, the conclusion is the same because the fibres of p_2 have constant dimension equal to $-\chi(N_1 \otimes I_{\xi_1}, N_2 \otimes I_{\xi_2}) + 1$ and the fibres of q_2 are smooth Quot-schemes of dimension 1. Indeed, $\dim \text{Hom}(N_1, N_1) = 1$ and $\text{Ext}^1(N_1, N_1) = 0$.

On the other hand, if $N_1 \simeq N_2$ and $l_2 \neq 0$, the dimension of the fibres of p_2 is not necessarily constant; indeed, $\dim \text{Hom}(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1})$ depends on the reciprocal position of ξ_1 and ξ_2 . Since $\mathcal{H}om(I_{\xi_2}, \mathcal{O}_S) \simeq \mathcal{H}om(I_{\xi_2}, I_{\xi_2}) \simeq \mathcal{O}_S$ (cf. [22]), one shows that

$$\mathcal{H}om(I_{\xi_2}, I_{\xi_1}) \simeq \{f \in \mathcal{O}_S \mid f \cdot I_{\xi_2} \subseteq I_{\xi_1}\} =: (I_{\xi_1} : I_{\xi_2}) = I_{\xi_1 \setminus (\xi_1 \cap \xi_2)};$$

hence, one finds that

$$\dim \text{Hom}(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1}) = H^0(S, \mathcal{H}om(I_{\xi_2}, I_{\xi_1})) = \begin{cases} 1 & \text{if } \xi_1 \subseteq \xi_2 \\ 0 & \text{otherwise} \end{cases}. \quad (3.27)$$

As in [26], define $\mathcal{N}_{N,N_2,l_2}^0$ (resp. $\mathcal{N}_{N,N_2,l_2}^1$) to be the substack of $\mathcal{M}(v_2) \times \mathcal{M}(v_1)$ whose points correspond to pairs $(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1})$ with $\xi_1 \not\subseteq \xi_2$ (resp. $\xi_1 \subseteq \xi_2$), i.e., $\dim \operatorname{Hom}(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1}) = 0$ (resp. $\dim \operatorname{Hom}(N_1 \otimes I_{\xi_2}, N_1 \otimes I_{\xi_1}) = 1$). Notice that $\mathcal{N}_{N,N_2,l_2}^0$ and $\mathcal{N}_{N,N_2,l_2}^1$ are complementary and that, being open, $\mathcal{N}_{N,N_2,l_2}^0$ is dense in $\mathcal{M}(v_2) \times \mathcal{M}(v_1)$ provided $l_1 \neq 0$.

We define $\mathcal{F}_{N,N_2,l_2}^0 := (p_2)^{-1}(\mathcal{N}_{N,N_2,l_2}^0)$ and $\mathcal{F}_{N,N_2,l_2}^1 := (p_2)^{-1}(\mathcal{N}_{N,N_2,l_2}^1)$ and we denote by Q_{N,N,l_2}^0 and Q_{N,N,l_2}^1 the closure of the image under q_2 of $\mathcal{F}_{N,N_2,l_2}^0$ and $\mathcal{F}_{N,N_2,l_2}^1$ respectively. Since the fibres of q_2 are Quot-schemes, we obtain that:

$$\begin{aligned} \dim Q_{N,N,l_2}^0 &= \dim \mathcal{F}_{N,N_2,l_2}^0 = \dim \mathcal{N}_{N,N_2,l_2}^0 + \langle v_1, v_2 \rangle \leq 2l_1 + 2l_2 - 2 + \langle v_1, v_2 \rangle, \\ \dim Q_{N,N,l_2}^1 &\leq \dim \mathcal{F}_{N,N_2,l_2}^1 = \dim \mathcal{N}_{N,N_2,l_2}^1 + \langle v_1, v_2 \rangle + 1 \leq 2l_1 + 2l_2 - 1 + \langle v_1, v_2 \rangle, \end{aligned}$$

where the last inequality in the second row is strict, unless the stack $\mathcal{N}_{N,N_2,l_2}^1$ is dense in $\mathcal{M}(v_2) \times \mathcal{M}(v_1)$, that is, $l_1 = 0$.

The statement follows because every component of Q_{N,N,l_2} is contained either in Q_{N,N,l_2}^0 or in Q_{N,N,l_2}^1 . \square

By proceeding as in Lemma 3.9.1, one proves the following:

Proposition 3.9.2. *Let Z be a nonempty irreducible component of P_{N,N_2,l_2} . We have that*

$$\dim Z = 2l_1 + 2l_2 + \langle v_2, v \rangle + \langle v_1, v \rangle + \langle v_1, v_2 \rangle - \alpha, \quad (3.28)$$

where α satisfies:

- (a) If N, N_1, N_2 are all non-isomorphic, then $\alpha = 3$.
- (b) Assume $N \simeq N_1 \simeq N_2$. If $l_2 \neq 0$ and $l_1 = 0$, then $\alpha \in \{1, 2, 3\}$. If $l_1 \neq 0$ and $l_2 = 0$, one has $\alpha \in \{2, 3\}$. In all the other cases, $\alpha = 3$. If $N \simeq N_1 \not\simeq N_2$, one has $\alpha = 3$ unless $l_1 = 0$, in which case $\alpha \in \{2, 3\}$.
- (c) If $N \simeq N_2 \not\simeq N_1$, then $\alpha = 3$ unless $l_2 = 0$, in which case $\alpha \in \{2, 3\}$.
- (d) Assume $N_1 \simeq N_2 \not\simeq N$. Then $\alpha = 3$ except when $l_2 \neq 0$ and $l_1 = 0$; in this case $\alpha \in \{2, 3\}$.

Note that LM bundles of type (i) lie in some P_{N,N_2,l_2} with N, N_1, N_2 as in case (a). Analogously, if E is a LM bundle of type (iii) (resp. of type (iv)), then there exist $N, N_2, N_1 = L \otimes (N \otimes N_2)^\vee$ as in (a) or (c) (resp. as in (a) or (d)) and $l_2 \in \mathbb{N}$ such that $[E] \in P_{N,N_2,l_2}(\mathbb{C})$. On the other hand, if a bundle of type (ii) defines a point of $P_{N,N_2,l}$, then $\mu_L(N) = \mu_L(N_2) = \mu_L(N_1)$ and any case of the previous proposition may occur.

Now, we consider the Grassmann bundle $\psi : \mathcal{G}_{N,N_2,l_2} \rightarrow P_{N,N_2,l_2}$ with fibre over a point $[E] \in P_{N,N_2,l_2}(\mathbb{C})$ equal to $G(3, H^0(S, E))$ and denote by \mathcal{W}_{N,N_2,l_2} the closure of the image of the rational map $h_{N,N_2,l_2} : \mathcal{G}_{N,N_2,l_2} \dashrightarrow \mathcal{W}_d^2(|L|)$.

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

Lemma 3.9.3. *Assume that general curves in $|L|$ have Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$. Then, for any irreducible component \mathcal{W} of \mathcal{W}_{N,N_2,l_2} , one has*

$$\dim \mathcal{W} \leq \frac{1}{4}g + d + \frac{3}{2} - \alpha,$$

where α is as in Proposition 3.9.2.

Proof. Let \mathcal{G} be an irreducible component of \mathcal{G}_{N,N_2,l_2} such that $\mathcal{W} = \overline{h_{N,N_2,l_2}(\mathcal{G})}$. Since $\mathcal{G} = \psi^{-1}(Z)$ for some irreducible component Z of P_{N,N_2,l_2} , Proposition 3.9.2 implies that:

$$\begin{aligned} \dim \mathcal{G} &= 3(g - d + 2) + \dim Z \\ &= 3(g - d) + 12 - \alpha - 2\chi(E) + 2l_1 + 2l_2 + \\ &\quad c_1(N) \cdot c_1(N_1) + c_1(N) \cdot c_1(N_2) + c_1(N_1) \cdot c_1(N_2) \\ &= g - d + 2 - \alpha + 2(l_1 + l_2) + c_1(N) \cdot (c_1(N_1) + c_1(N_2)) + c_1(N_1) \cdot c_1(N_2) \\ &= g + d + 2 - \alpha - c_1(N) \cdot c_1(N_1) - c_1(N) \cdot c_1(N_2) - c_1(N_1) \cdot c_1(N_2) \\ &\leq g + d + 2 - \alpha - \frac{3}{2}k \\ &\leq \frac{1}{4}g + d + \frac{1}{2} - \alpha, \end{aligned}$$

where we have used Lemma 3.6.3 and the fact that $k \geq (g + 2)/2$. The statement follows since the fibres of h_{N,N_2,l_2} are quotient stacks of dimension -1 . \square

Finally, we prove the following:

Proposition 3.9.4. *Assume that general curves in $|L|$ have Clifford dimension 1 and maximal gonality $k = \left\lfloor \frac{g+3}{2} \right\rfloor$. Fix a positive integer d such that $(g, d) \notin \{(2, 4), (4, 5), (6, 6), (10, 9)\}$. Let $\mathcal{W} \subset \mathcal{W}_{N,N_2,l_2}$ be an irreducible component of $\mathcal{W}_d^2(|L|)$. Then $\rho(g, 2, d) \geq 0$ and \mathcal{W} does not dominate $|L|$.*

Proof. Lemma 3.6.3 implies $d \geq \frac{3}{2}k$, hence $\rho(g, 2, d) \geq 0$. Lemma 3.9.3 gives:

$$\dim \mathcal{W} \leq \frac{1}{4}g + d + \frac{3}{2} - \alpha.$$

Therefore, \mathcal{W} cannot dominate $|L|$ if

$$\frac{1}{4}g + d + \frac{3}{2} - \alpha < g + \rho(g, 2, d) = -g + 3d - 6,$$

that is, $d > \frac{5}{8}g + \frac{15}{4} - \frac{\alpha}{2}$. In particular, as $\alpha \geq 1$, it suffices to require $d > \frac{5}{8}g + \frac{13}{4} =: h$. Such inequality is satisfied always except for

$$(g, d) \in \{(2, 4), (3, 5), (4, 5), (5, 6), (6, 6), (6, 7), (8, 8), (10, 9), (14, 12)\}.$$

If $(g, d) = (6, 6)$, the linear system $|L|$ can be dominated by \mathcal{W} . In all the other cases $d = \lfloor h \rfloor$ and we check whether $\alpha > 2h - 2 \lfloor h \rfloor + 1$, which would prevent \mathcal{W} from being dominant. This holds true if $(g, d) \notin \{(2, 4), (4, 5), (10, 9)\}$ (use that the case $\alpha = 1$ may occur only when parametrizing LM bundles of type (ii) and that, if $\gcd(2g - 2, 3) = 1$, there do not exist properly μ_L -semistable of Mukai vector $v(E)$). \square

Remark 3. The four cases which are not covered by Proposition 3.9.4 might be treated by “ad hoc” arguments but this is not our purpose.

Proofs of Theorem 3.1.1 and Theorem 3.1.2 are now straightforward.

Proof of Theorem 3.1.1. Being non-simple, the LM bundle $E_{C,A}$ is not μ_L -stable. Since $d < \frac{2}{3}g + 2$, Corollary 3.6.4 implies the existence of a line bundle $N \in \text{Pic}(S)$ such that $E_{C,A} \in R_N(\mathbb{C})$. The statement thus follows directly from Proposition 3.8.4. \square

Proof of Theorem 3.1.2. Case (a) trivially follows from Proposition 3.7.3, Proposition 3.8.3 and Proposition 3.9.4.

Now, let $\frac{2}{3}g + 2 \leq d \leq \frac{3}{4}g + 2$. Given \mathcal{W} an irreducible component of $\mathcal{W}_d^2(|L|)$ which dominates $|L|$ and whose general point corresponds to a LM bundle that is not μ_L -stable, Proposition 3.7.3 and Proposition 3.9.4 imply the existence of a line bundle $N \in \text{Pic}(S)$ such that $\mathcal{W} \subset \mathcal{W}_N$. The statement follows from Proposition 3.8.4. \square

3.10 Transversality of some Brill-Noether loci

We apply our results in order to prove Theorem 3.1.4 in the introduction.

Theorem 3.10.1. *Let $r \geq 3$, $g \geq 0$, $d \leq g - 1$ be positive integers such that $\rho(g, r, d) < 0$ and $d - 2r + 2 \geq \lfloor (g + 3)/2 \rfloor$. If $r \geq 4$, assume $d^2 > 4(r - 1)(g + r - 2)$. For $r = 3$, let $d^2 > 8g + 1$. If -1 is not represented by the quadratic form*

$$Q(m, n) = (r - 1)m^2 + mnd + (g - 1)n^2 \quad m, n \in \mathbb{Z},$$

there exists a smooth curve $C \subset \mathbb{P}^r$ of genus g , degree d and maximal gonality $\lfloor \frac{g+3}{2} \rfloor$. Moreover, one can choose C such that for any complete, base point free g_e^1 on C with $\rho(g, 1, e) \geq 0$ the Petri map is injective.

Proof. Notice that the inequalities $d \leq g - 1$ and $d^2 > 4(r - 1)(g - 1)$ trivially imply $d > 4(r - 1)$.

In order to prove the first part of the statement, we proceed as in [6, Theorem 3] paying special attention to our slightly different hypotheses. Rathmann’s Theorem implies the existence of a $2r - 2$ -degree K3 surface $S \subset \mathbb{P}^r$ and a smooth curve $C \subset S$ of degree d and genus g such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where H is the hyperplane section of S . Our assumption on Q implies that S does not contain (-2) -curves. As in [6], one shows that the line bundle $L := \mathcal{O}_S(C)$ is ample by Nakai-Moishezon criterion (if $D \subset S$ is

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

an effective divisor, use that $D^2 \geq 0$ and $D \cdot H > 2$, in order to show that $C \cdot D > 0$. Hence, C has Clifford dimension 1 (cf. [4] Proposition 3.3).

Assume that C has gonality $k < \left\lfloor \frac{g+3}{2} \right\rfloor$. We reach a contradiction by showing that $k \geq d - 2r + 2$. If A is a complete, base point free pencil g_k^1 on C , by [5] Theorem (4.2) there exists an effective divisor $D \equiv mH + nC$ on S , such that $|A|$ is contained in the linear system $|\mathcal{O}_C(D)|$ and the following conditions are satisfied:

$$h^0(S, \mathcal{O}_S(D)) \geq 2, \quad h^0(S, \mathcal{O}_S(C - D)) \geq 2, \quad C \cdot D \leq g - 1, \quad \text{Cliff}(D|_C) = \text{Cliff}(A).$$

In particular, as remarked in [5, page 60], the last equality implies that

$$h^1(S, \mathcal{O}_S(D)) = h^1(S, \mathcal{O}_S(C - D)) = 0,$$

thus $c_1(D)^2 > 0$ and $c_1(C - D)^2 > 0$. Moreover, one has

$$k = 2 + \text{Cliff}(D|_C) = D \cdot (C - D).$$

We show that

$$f(m, n) = D \cdot C - D^2 = -(2r - 2)m^2 + d(1 - 2n)m + (n - n^2)(2g - 2) \geq d - 2r + 2,$$

for values of m and n satisfying the following inequalities:

- (i) $(r - 1)m^2 + mnd + n^2(g - 1) > 0$,
- (ii) $(r - 1)m^2 + (mn - m)d + (1 - n)^2(g - 1) > 0$,
- (iii) $2 < (2r - 2)m + nd < d - 2$,
- (iv) $md + (2n - 1)(g - 1) \leq 0$.

Assume first $n = 1$, and set $a = -m$. Then (iii) implies $0 < a < (d - 2)/(2r - 2)$. Inequality (i) is equivalent to $(r - 1)a^2 - ad + g - 2 \geq 0$, whence

$$a \leq \frac{d - \sqrt{d^2 - 4(r - 1)(g - 2)}}{2r - 2}.$$

We have $f(-a, 1) \geq d - 2r + 2$ whenever $1 \leq a \leq d/(2r - 2) - 1$. For either $r \geq 4$ or $r = 3$ and $d^2 - 8g \geq 8$, this holds true as $d^2 - 4(r - 1)(g - 2) \geq 4r(r - 1) > 4(r - 1)^2$. If $r = 3$ and $d^2 - 8g < 8$, then $d^2 - 8g = 4$ and $d \equiv 2 \pmod{4}$. Hence, (iii) implies that $1 \leq a < (d - 4)/4$. Notice that $f(-a, 1) = d - 2r + 2$ whenever $a = 1$, that is, $C \equiv C - H$. The case $n = 0$ can be treated similarly by using (ii) instead of (i), and one obtains that $f(m, 0) \geq d - 2r + 2$ with equality holding only for $m = 1$, that is, $D \equiv H$.

If $n < 0$, inequalities (i), (iii) and (iv) imply that $-\alpha n < m \leq (g - 1)(1 - 2n)/d$, where

$$\alpha = \frac{d + \sqrt{d^2 - 4(r - 1)(g - 1)}}{2r - 2}.$$

Therefore, one has

$$f(m, n) \geq \min \left\{ f(-\alpha n, n), f\left(\frac{(g-1)(1-2n)}{d}, n\right) \right\}.$$

Analogously, if $n \geq 2$, then $\max\{-\beta n, (2-nd)/(2r-2)\} < m \leq (g-1)(1-2n)/d$, where

$$\beta = \frac{d - \sqrt{d^2 - 4(r-1)(g-1)}}{2r-2};$$

this gives

$$f(m, n) \geq \min \left\{ f\left(\frac{(g-1)(1-2n)}{d}, n\right), \max \left\{ f(-\beta n, n), f\left(\frac{2-nd}{2r-2}, n\right) \right\} \right\}.$$

Computations in [6] give $\max\{f(-\beta n, n), f((2-nd)/(2r-2), n)\} > d-2r+2$ when $n \geq 2$, and $f(-\alpha n, n) > d-2r+2$ when $n < 0$, unless $r = 3$, $n = -1$ and $d^2 - 8g = 4$. In this case, $d \equiv 2 \pmod{4}$ and $m \geq (d+4)/4$ by (iii); one uses that $f((d+4)/4, -1)$ is greater than $d-4$. In order to conclude the proof that C has maximal gonality, it is enough to remark that the function

$$h(n) := f\left(\frac{(g-1)(1-2n)}{d}, n\right) = \frac{g-1}{2} \left[\frac{(2n-1)^2(d^2 - 4(r-1)(g-1))}{d^2} + 1 \right]$$

reaches its minimum for $n = 1/2$ and $h(0) \geq d-2r+2$ by direct computation.

Concerning the last part of the statement, assume C is general in its linear system and let A be a complete, base point free pencil g_e^1 on C such that $\rho(g, 1, e) \geq 0$ and $\ker \mu_{0,A} \neq 0$. The bundle $E = E_{C,A}$ is non-simple, hence it cannot be μ_L -stable. As a consequence, there exists a short exact sequence

$$0 \rightarrow M \rightarrow E \rightarrow N \otimes I_{\xi} \rightarrow 0, \quad (3.29)$$

where M, N are line bundles, I_{ξ} is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$ and $c_1(M) \cdot C \geq \mu_L(E) = g-1 \geq c_1(N) \cdot C$. If sequence (3.29) does not split, then

$$h^0(S, E \otimes E^{\vee}) \leq 1 + \dim \operatorname{Hom}(M, N \otimes I_{\xi}) + \dim \operatorname{Hom}(N \otimes I_{\xi}, M).$$

Since $\mu_L(M) \geq \mu_L(N)$, if $\operatorname{Hom}(M, N \otimes I_{\xi}) \neq 0$ then $M \simeq N$ and $C = 2c_1(M)$, which is absurd. It follows that $N^{\vee} \otimes M$ is non-trivial and effective. Since S does not contain (-2) -curves, one has

$$c_1(N^{\vee} \otimes M)^2 = C^2 - 4c_1(N) \cdot c_1(M) = 2g - 2 - 4c_1(N) \cdot c_1(M) \geq 0;$$

this contradicts Lemma 3.5.1, which states that $c_1(N) \cdot c_1(M) \geq k \geq (g+2)/2$. Thus, $\xi = \emptyset$ and sequence (3.29) splits. We have to show that, if $E = N \oplus M$ is a splitting LM bundle, the rational map $\chi_E : G(2, H^0(S, E)) \dashrightarrow |L|$ cannot be dominant.

Notice that χ_E factors through $h_E : G(2, H^0(S, E)) \dashrightarrow \mathcal{W}_e^1(|L|)$; the fibre of h_E over a

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

point $(C, A) \in \text{Im } h_E$ is at least 1-dimensional since it is isomorphic to

$$\mathbb{P}(\text{Hom}(E_{C,A}, \omega_C \otimes A^\vee)^\circ),$$

where $\text{Hom}(E_{C,A}, \omega_C \otimes A^\vee)^\circ$ is an open subgroup of

$$\text{Hom}(E_{C,A}, \omega_C \otimes A^\vee) \simeq H^0(S, E_{C,A} \otimes E_{C,A}^\vee).$$

We are done, as $\rho(g, 1, e) \geq 0$, hence $\dim G(2, H^0(S, E)) = 2(g - e + 1) \leq g$. \square

Theorem 3.10.2. *Let g, r, d satisfy the hypotheses of Theorem 3.10.1. The curve C can be chosen so that, if*

$$e < \min \left\{ d - 2r + 5, \frac{17}{24}g + \frac{23}{12} \right\},$$

then C does not have any complete, base point free net g_e^2 for which the Petri map is non-injective.

Proof. Let $S \subset \mathbb{P}^r$ be a K3 surface as in the proof of Theorem 3.10.1 and $C \subset S$ be general in its linear system. Let A be a complete, base point free $g_{d_A}^2$ on C with $d_A < \frac{17}{24}g + \frac{23}{12}$; if $\rho(g, 2, d_A) \geq 0$, assume moreover that $\ker \mu_{0,A} \neq 0$. Corollary 3.6.4 and Proposition 3.8.4 imply that $|A|$ is contained in the linear system $|\mathcal{O}_C(D)|$ for some effective divisor $D \equiv mH + nC$ on S such that:

$$h^0(S, \mathcal{O}_S(D)) \geq 2, \quad h^0(S, \mathcal{O}_S(C - D)) \geq 2, \quad C \cdot D \leq \frac{4g - 4}{3}, \quad \text{Cliff}(D|_C) \leq \text{Cliff}(A).$$

In fact, the Lazarsfeld-Mukai bundle $E := E_{C,A}$ is given by an extension

$$0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0,$$

where $N := \mathcal{O}_S(C - D)$ and E/N is a μ_L -stable torsion free sheaf of rank 2 on S . As in the proof of Proposition 3.8.4, one shows that $D^2 > 0$, hence $h^1(S, \mathcal{O}_S(D)) = 1$. Moreover, one obtains that $h^1(S, N) = 0$ because the equality $(C - D)^2 = 0$ would imply $d \geq (5g + 4)/6$, which is absurd. As a consequence, one has

$$d_A - 4 = \text{Cliff}(A) \geq \text{Cliff}(D|_C) = D \cdot C - 2h^0(S, \mathcal{O}_C(D|_C)) + 2 = D \cdot (C - D) - 2, \quad (3.30)$$

and equality holds whenever $D^2 = 2$ and $c_2(E/N) = 2$ (cf. proof of Proposition 3.8.4); in particular, for $D \equiv H$, the inequality is strict. We show that

$$f(m, n) := D \cdot (C - D) \geq d - 2r + 2, \quad (3.31)$$

and, if equality holds, then either $D \equiv H$ or $D \equiv C - H$. Computations are similar to those in Theorem 3.10.1, but now, instead of having $D \cdot C \leq g - 1$, we only know that $D \cdot C \leq (4g - 4)/3$. Therefore, inequality (iv) must be replaced with

$$(iv') \quad md + (2n - \frac{4}{3})(g - 1) \leq 0.$$

The cases $n \in \{0, 1\}$ can be treated exactly as before. For $n < 0$, we have

$$f(m, n) \geq \min \left\{ f(-\alpha n, n), f\left(\frac{(g-1)(\frac{4}{3}-2n)}{d}, n\right) \right\}.$$

If $n \geq 2$, then

$$f(m, n) \geq \min \left\{ f\left(\frac{(g-1)(\frac{4}{3}-2n)}{d}, n\right), \max \left\{ f(-\beta n, n), f\left(\frac{2-nd}{2r-2}, n\right) \right\} \right\}.$$

Therefore, it is enough to show that

$$g(n) := f\left(\frac{(g-1)(\frac{4}{3}-2n)}{d}, n\right) - d + 2r - 2 > 0 \text{ for } n < 0 \text{ or } n \geq 2.$$

One can write $g(n) = an^2 + bn + c$, with

$$\begin{aligned} a &= -4(2r-2) \left(\frac{g-1}{d}\right)^2 + 2d \left(\frac{g-1}{d}\right), \\ b &= \frac{16}{3}(2r-2) \left(\frac{g-1}{d}\right)^2 - \frac{8}{3}d \left(\frac{g-1}{d}\right), \\ c &= -\frac{16}{9}(2r-2) \left(\frac{g-1}{d}\right)^2 + \frac{4}{3}d \left(\frac{g-1}{d}\right) - d + 2r - 2. \end{aligned}$$

Since $a > 0$ and $0 < -b/2a < 1$, our claim follows if $g(0) = c > 0$, or equivalently, if

$$\frac{3}{4} < \frac{g-1}{d} < \frac{3}{8} \left(\frac{d-2(r-1)}{r-1}\right).$$

The left inequality is trivial since $d \leq g-1$. The right inequality is equivalent to the condition $8(g-1)(r-1) < 3d^2 - 6d(r-1)$, which is satisfied as well (if $r \geq 4$, use that $8(g-1)(r-1) < 2d^2 - 8(r-1)^2$ and $d > 4(r-1)$; if $r = 3$, use that $d^2 > 8g + 1$ and either $(g, d) = (12, 11)$ or $d \geq 12$ by manipulation of the hypotheses).

We conclude that $d_A \geq d - 2r + 4$ and the inequality is strict unless equalities hold both in (3.30) and (3.31), thus $D \equiv C - H$ and $(C - H)^2 = 2$. This case can be excluded since it would imply $d = g + r - 3 \geq g$. \square

Note that the condition $e < \frac{17}{24}g + \frac{23}{12}$ is automatically satisfied if $\rho(g, 2, e) < 0$.

The proof of Theorem 3.1.4 is now trivial: apply Theorem 3.10.1 and Theorem 3.10.2 and proceed by induction on f and e in order to deal with pencils g_f^1 and nets g_e^2 which have a nonempty base locus.

3.11 Noether-Lefschetz divisor and Gieseker-Petri divisor in genus 11

The Clifford index $\text{Cliff}(C)$ is one of the most important invariants of an algebraic curve C . In [14] Lange and Newstead defined the analogue of the Clifford index for higher rank vector bundles in the following way. If $\mathcal{U}_C(n, d)$ denotes the moduli space of semistable rank- n vector bundles of degree d on a genus- g curve C , given $E \in \mathcal{U}_C(n, d)$, the Clifford index of E is

$$\gamma(E) := \mu(E) - \frac{2}{n}h^0(C, E) + 2 \geq 0,$$

where $\mu(E)$ denotes the slope of E . For any positive integer n , one defines the higher Clifford index of C

$$\text{Cliff}_n(C) := \min\{\gamma(E) \mid E \in \mathcal{U}_C(n, d), h^0(C, E) \geq 2n, \mu(E) \leq g - 1\}.$$

A natural question is whether higher Clifford indices are new invariants, different from the ones arising in classical Brill-Noether theory. In [14] Lange and Newstead reformulated a conjecture of Mercat (cf. [19]) in a slightly weaker form predicting:

$$\text{Cliff}_n(C) = \text{Cliff}(C); \tag{3.32}$$

remark that trivially $\text{Cliff}_n(C) \leq \text{Cliff}(C)$, while the opposite inequality is largely non-trivial. When $n = 2$, the conjecture has been proved for a general curve in M_g , if $g \leq 16$, by Farkas and Ortega (cf. [8]) and the same is expected to hold true in any genus. However, if $g \geq 11$, there are examples of curves with maximal Clifford index $\text{Cliff}(C) = \left\lfloor \frac{g-1}{2} \right\rfloor$ that violate (3.32) for $n = 2$. These have been constructed in [8], [7], [14], [15], [13] as sections of K3 surfaces with Picard number at least 2. We recall that the K3 locus

$$\mathcal{K}_g := \{[C] \in M_g \mid C \subset S, S \text{ is a K3 surface}\}$$

is irreducible of dimension $19 + g$ if either $g = 11$ or $g \geq 13$ (cf. [3]). In particular, $\mathcal{K}_{11} = M_{11}$, and a general curve $[C] \in M_{11}$ lies on a unique K3 surface with Picard number one (cf [21]). Given two positive integers r, d such that $d^2 > 4(r - 1)g$ and d does not divide $2r - 2$, one defines the Noether-Lefschetz divisor inside \mathcal{K}_g as

$$\mathcal{NL}_{g,d}^r := \left\{ [C] \in \mathcal{K}_g \mid \begin{array}{l} C \subset S, S \text{ is a K3 surface, } \text{Pic}(S) \supset \mathbb{Z}C \oplus \mathbb{Z}H, \\ H \text{ nef, } H^2 = 2r - 2, C^2 = 2g - 2, C \cdot H = d \end{array} \right\}.$$

In [7] it is proved that a curve C of genus 11 violates Mercat's conjecture for $n = 2$, whenever $[C] \in \mathcal{NL}_{11,13}^4$.

Since some of the curves exhibited in [14], [15], [13] do not satisfy the Gieseker-Petri Theorem, Lange and Newstead asked whether $\text{Cliff}_2(C) = \text{Cliff}(C)$ whenever C is a Petri curve (Question 4.2 in [13]). We prove Theorem 3.1.5, which gives a negative

answer to this question.

Let $S \subset \mathbb{P}^4$ be a K3 surface such that $\text{Pic}(S) = \mathbb{Z}C \oplus \mathbb{Z}H$, where H is the hyperplane section, $H^2 = 6$, $C^2 = 20$ and $C \cdot H = 13$. Denote by L the line bundle $\mathcal{O}_S(C)$. We show that, if $C \in |L|$ is general, then $[C]$ does not lie in the Gieseker-Petri locus GP_{11} . As proved in Chapter 2, GP_{11} has pure codimension 1 in M_{11} and decomposes in the following way:

$$GP_{11} = M_{11,9}^2 \cup GP_{11,10}^2 \cup \bigcup_{d=7}^{10} GP_{11,d}^1,$$

where $M_{11,9}^2$ is a Brill-Noether divisor. Therefore, proving the transversality of $\mathcal{NL}_{11,13}^4$ and GP_{11} is equivalent to showing that in the above situation, if $C \in |L|$ is general, then C has no g_9^2 and the varieties $G_{10}^2(C)$ and $G_d^1(C)$ for $7 \leq d \leq 10$ are smooth of the expected dimension.

We proceed as in the previous section; since the hypotheses of Theorem 3.10.1 are not satisfied, explicit computations must be performed. Direct computations imply that S contains no (-2) -curves. Moreover, C is an ample line bundle on S by [13, Proposition 2.1]. As a consequence, C has Clifford dimension 1 (cf. [4, Proposition 3.3]) and $\text{Cliff}(C) = 5$ (cf. [7, Proposition 3.3]). In particular, C has maximal gonality $k = 7$ and has no g_d^2 for $d \leq 8$. Hence, in order to prove that $G_9^2(C) = \emptyset$, it is enough to exclude the existence of complete, base point free g_9^2 on C . Similarly, the condition $[C] \notin GP_{11,10}^2$ is equivalent to the requirement for $G_{10}^2(C)$ to be smooth of the expected dimension $\rho(11, 2, 10)$ in the points corresponding to complete, base point free linear series. Analogously, by induction on d , if the Petri map associated with any complete, base point free pencil of degree $7 \leq d \leq 10$ is injective, then $[C] \notin \bigcup_{d=7}^{10} GP_{11,d}^1$.

For any $A \in G_9^2(C)$, the Petri map $\mu_{0,A}$ is non-injective for dimensional reasons and the bundle $E = E_{C,A}$ is non-simple, hence it cannot be μ_L -stable. Since

$$\gcd(\text{rk } E, c_1(E)^2) = \gcd(3, 20) = 1,$$

there are no properly semistable sheaves of Mukai vector $v(E) = (3, C, 4)$; hence, E is μ_L -unstable. By Corollary 3.6.4, E sits in the short exact sequence

$$0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0, \quad (3.33)$$

where $N \in \text{Pic}(S)$ coincides with its maximal destabilizing sheaf and the quotient E/N is a μ_L -stable torsion free sheaf of rank 2. Having denoted by D the first Chern class of E/N , Proposition 3.8.4 implies that the line bundle $\mathcal{O}_C(D)$ contributes to $\text{Cliff}(C)$. Moreover, as in the proof of the aforementioned proposition, one shows that

$$D^2 \geq 2, \quad (3.34)$$

$$c_2(E/N) \geq \frac{3}{2} + \frac{1}{4}D^2. \quad (3.35)$$

3 Stability of rank-3 Lazarsfeld-Mukai bundles on K3 surfaces

Furthermore, Lemma 3.6.2 gives

$$c_1(N) \cdot c_1(E/N) = (C - D) \cdot D \geq k = 7. \quad (3.36)$$

Since

$$9 = c_2(E) = c_2(E/N) + (C - D) \cdot D \geq \frac{3}{2} + \frac{1}{4}D^2 + (C - D) \cdot D,$$

the divisor $D \equiv mH + nC$ must satisfy

$$\begin{cases} C \cdot D = 13m + 20n = 9 \\ D^2 = 6m^2 + 20n^2 + 26mn = 2(m + n)(3m + 10n) = 2 \end{cases}.$$

One shows that this system admits no integral solution. As a consequence, a general curve in $|L|$ has no linear series of type g_9^2 .

Analogously, given a complete, base point free $A \in G_{10}^2(C)$ with $\ker \mu_{0,A} \neq 0$, the LM bundle $E = E_{C,A}$ is μ_L -unstable and its maximal destabilizing sheaf is a line bundle N such that E/N is μ_L -stable by Corollary 3.6.4. With the same notation as above, inequalities (3.34), (3.35), (3.36) still hold true and the following cases must be considered:

$$\begin{aligned} (a) \quad & \begin{cases} C \cdot D = 10 \\ D^2 = 2 \\ (c_2(E/N) = 2) \end{cases} & (b) \quad & \begin{cases} C \cdot D = 9 \\ D^2 = 2 \\ (c_2(E/N) = 3) \end{cases} \\ (c) \quad & \begin{cases} C \cdot D = 11 \\ D^2 = 4 \\ (c_2(E/N) = 3) \end{cases} & (d) \quad & \begin{cases} C \cdot D = 13 \\ D^2 = 6 \\ (c_2(E/N) = 3) \end{cases}. \end{aligned}$$

These systems have no integral solutions except for (d), which is satisfied by

$$(m, n) = (1, 0).$$

Therefore, $N = \mathcal{O}_S(C - H)$ and $v(E/N) = (2, H, 2)$. As $\langle v(E/N), v(E/N) \rangle = -2$, the sheaf E/N is uniquely determined.

By applying first $\text{Hom}(E, -)$ and then $\text{Hom}(-, N)$ and $\text{Hom}(-, E/N)$ to the short exact sequence (3.33), one shows that

$$h^0(S, E \otimes E^\vee) \leq 2 + \dim \text{Hom}(N, E/N) + \dim \text{Hom}(E/N, N)$$

and the inequality is strict if the sequence does not split. As $\mu_L(N) > \mu_L(E/N)$, Proposition 3.4.1 implies that $\text{Hom}(N, E/N) = 0$. Let $0 \neq \alpha \in \text{Hom}(E/N, N)$. Since both $\text{Im } \alpha$ and $\ker \alpha$ are torsion free sheaves of rank 1, there exists an effective divisor D_1 on S and two 0-dimensional subschemes $\xi_1, \xi_2 \subset S$ such that E/N is given by an extension

$$0 \rightarrow \mathcal{O}_S(2H - C + D_1) \otimes I_{\xi_1} \rightarrow E/N \rightarrow \mathcal{O}_S(C - H - D_1) \otimes I_{\xi_2} \rightarrow 0.$$

3.11 Noether-Lefschetz divisor and Gieseker-Petri divisor in genus 11

The μ_L -stability of E/N implies that

$$13/2 = \mu_L(E/N) < (C - H - D_1) \cdot C = -D_1 \cdot C + 7;$$

since C has positive intersection with any non-trivial effective divisor, $D_1 = 0$. It follows that

$$3 = c_2(E/N) = (2H - C) \cdot (C - H) + l(\xi_1) + l(\xi_2) \geq 7,$$

which is absurd. Hence, $\text{Hom}(E/N, N) = 0$ and (3.33) splits. As a consequence, the bundle $E = N \oplus E/N$ is uniquely determined.

We look at the rational map $\chi : G(3, H^0(S, E)) \dashrightarrow |L|$; this cannot be dominant since $\dim G(3, H^0(S, E)) = 9$. Therefore, a general curve $C \in |L|$ does not lie in $GP_{11,10}^2$.

It remains to show that, if $C \in |L|$ is general, then $[C] \notin \cup_{d=7}^{10} GP_{11,d}^1$. It is enough to prove that for any complete, base point free g_d^1 on C the Petri map is injective. One can proceed exactly as in the last part of the proof of Theorem 3.10.1 since S does not contain (-2) -curves.

Bibliography

- [1] M. Aprodu and G. Farkas. Green's conjecture for curves on arbitrary $K3$ surfaces. *Compos. Math.*, 147(3):839–851, 2011.
- [2] T. Bridgeland. An introduction to motivic hall algebras. [ArXiv:1002.4372](#).
- [3] C. Ciliberto, A. Lopez, and R. Miranda. Projective degenerations of $K3$ surfaces, Gaussian maps, and Fano threefolds. *Invent. Math.*, 114(3):641–667, 1993.
- [4] C. Ciliberto and G. Pareschi. Pencils of minimal degree on curves on a $K3$ surface. *J. Reine Angew. Math.*, 460:15–36, 1995.
- [5] R. Donagi and D. R. Morrison. Linear systems on $K3$ -sections. *J. Differ. Geom.*, 29(1):49–64, 1989.
- [6] G. Farkas. Brill-Noether loci and the gonality stratification of \mathcal{M}_g . *J. Reine Angew. Math.*, 539:185–200, 2001.
- [7] G. Farkas and A. Ortega. Higher rank brill-noether theory on sections of $k3$ surfaces. To appear.
- [8] G. Farkas and A. Ortega. The maximal rank conjecture and rank two brill-noether theory. *Pure Appl. Math. Q.*, 7:1265–1296, 2011.
- [9] R. Friedman. *Algebraic surfaces and holomorphic vector bundles*. Universitext . New York: Springer, 1998.
- [10] T. L. Gomez. Algebraic stacks. preprint.
- [11] M. Green and R. Lazarsfeld. Special divisors on curves on a $K3$ surface. *Invent. Math.*, 89:357–370, 1987.
- [12] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. 2nd ed. Cambridge: Cambridge University Press, 2010.
- [13] H. Lange and P. E. Newstead. Bundles of rank 2 with small clifford index on algebraic curves. [ArXiv:1105.4367](#).
- [14] H. Lange and P. E. Newstead. Clifford indices for vector bundles on curves. In *Affine Flag Manifolds and Principal Bundles* (A. H.W. Schmitt editor), *Trends in Mathematics*. Basel: Birkhäuser, 2010.

Bibliography

- [15] H. Lange and P. E. Newstead. Further examples of stable bundles of rank 2 with 4 sections. *Pure Appl. Math. Q.*, 7:1517–1538, 2011.
- [16] R. Lazarsfeld. Brill-Noether-Petri without degenerations. *J. Differ. Geom.*, 23:299–307, 1986.
- [17] R. Lazarsfeld. A sampling of vector bundle techniques in the study of linear series. In *Lectures on Riemann surfaces (Trieste, 1987)*, pages 500–559. World Sci. Publ., Teaneck, NJ, 1989.
- [18] A. L. Mayer. Families of K-3 surfaces. *Nagoya Math. J.*, 48:1–17, 1972.
- [19] V. Mercat. Clifford’s theorem and higher rank vector bundles. *Int. J. Math.*, 13(7):785–796, 2002.
- [20] S. Mukai. On the moduli space of bundles on K3 surfaces. I. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 341–413. Tata Inst. Fund. Res., Bombay, 1987.
- [21] S. Mukai. Curves and K3 surfaces of genus eleven. In *Moduli of vector bundles, Lecture Notes in Pure and Applied Math.*, volume 179. New York, NY: Marcel Dekker, 1996.
- [22] C. Okonek, M. Schneider, and H. Spindler. *Vector bundles on complex projective spaces*. Birkhauser, 1980.
- [23] G. Pareschi. A proof of Lazarsfeld’s theorem on curves on K3 surfaces. *J. Algebr. Geom.*, 4(1):195–200, 1995.
- [24] B. Saint-Donat. Projective models of K-3 surfaces. *Am. J. Math.*, 96:602–639, 1974.
- [25] S. S. Shatz. The decomposition and specialization of algebraic families of vector bundles. *Compos. Math.*, 35:163–187, 1977.
- [26] K. Yoshioka. Irreducibility of moduli spaces of vector bundles on k3 surfaces. [ArXiv:math/9907001v2](https://arxiv.org/abs/math/9907001v2).

4 Green's Conjecture for curves on Del Pezzo surfaces

4.1 Introduction

Syzygies of canonical curves are a subject of great interest in algebraic geometry. Two very classical results are Noether's Theorem, stating that a nonhyperelliptic curve C is embedded in \mathbb{P}^{g-1} by its canonical bundle ω_C as a projectively normal curve, and the Enriques-Babbage Theorem, asserting that the ideal of C in \mathbb{P}^{g-1} is generated by quadrics unless C is either trigonal or isomorphic to a plane quintic. Green's Conjecture was first stated in [9] and proposes a generalization of these results in terms of Koszul cohomology, predicting that

$$K_{p,2}(C, \omega_C) = 0 \text{ if and only if } p < \text{Cliff}(C). \quad (4.1)$$

Quite remarkably, this would imply that the Clifford index of C can be read off the syzygies of its canonical embedding. The implication $K_{p,2}(C, \omega_C) \neq 0$ for $p \geq \text{Cliff}(C)$ was immediately achieved by Green and Lazarsfeld ([9, Appendix]) and the conjectural part reduces to the vanishing $K_{c-1,2}(C, \omega_C) = 0$ for $c = \text{Cliff}(C)$, or equivalently, $K_{g-c-1,1}(C, \omega_C) = 0$.

One naturally expects the gonality k of C to be also encoded in the vanishing of some Koszul cohomology groups. In fact, Green-Lazarsfeld's Gonality Conjecture predicts that any line bundle A of sufficiently high degree on C satisfies

$$K_{p,1}(C, A) = 0 \text{ if and only if } p \geq h^0(C, A) - k. \quad (4.2)$$

Green ([9]) and Ehbauer ([8]) have shown that the statement holds true for any curve of gonality $k \leq 3$. As in the case of Green's Conjecture, one implication is well-known (cf. [9, Appendix]); it was proved by Aprodu (cf. [1]) that the conjecture is thus equivalent to the existence of a non-special globally generated line bundle A such that $K_{h^0(C,A)-k,1}(C, A) = 0$.

Both Green's Conjecture and Green-Lazarsfeld's Gonality Conjecture are in their full generality still open. However, as recalled in Section 1.7, by specialization to curves on K3 surfaces, they were proved for a general curve in each gonality stratum of M_g (cf. [14, 15, 2]). In the case of odd genus $g = 2k - 3$, the two conjectures are known to hold for every curve of maximal gonality k . Furthermore, it was proved by Aprodu that both Green's Conjecture and Green-Lazarsfeld's Gonality Conjecture are true for

4 Green's Conjecture for curves on Del Pezzo surfaces

every curve C of gonality $k \leq (g + 2)/2$ satisfying the *linear growth condition*

$$\dim W_d^1(C) \leq d - k \text{ for } k \leq d \leq g - k + 2. \quad (4.3)$$

In this Chapter we prove the following:

Theorem 4.1.1. *Let C be a smooth, irreducible curve lying on a Del Pezzo surface S and, having set $L := \mathcal{O}_S(C)$, assume that $L \otimes \omega_S$ is nef and big. Then, the following hold:*

- *If $\deg(S) \geq 2$, then C satisfies Green's Conjecture.*
- *If C is general in its linear system and $\text{Cliffdim}(C) = 1$, then C verifies Green-Lazarsfeld's Gonality Conjecture.*
- *If $\deg(S) = 1$, Green's Conjecture is true for a general curve in $|L|$; under the further assumption that the Clifford index of a general curve in $|L|$ is not computed by the restriction of the anticanonical bundle ω_S^\vee , Green's Conjecture holds for every smooth irreducible curve in $|L|$.*

Green's Conjecture for curves lying on arbitrary K3 surfaces was proved by Aprodu and Farkas in [3] (see Section 1.7 for details). Linear systems on Del Pezzo surfaces have some common behavior with those on K3 surfaces but, since Del Pezzo surfaces in general are non-minimal and their canonical bundle is non-trivial, their study is somehow more complicated; we will recall some known results in Section 4.3. Notice that the classes of line bundles L such that $L \otimes \omega_S$ is nef and big have been precisely described in [7] in terms of the coefficients of the generators of $\text{Pic}(S)$ in the presentation of L .

The Brill-Noether theory of curves lying on Del Pezzo surfaces was investigated by Pareschi (cf. [12]) and Knutsen (cf. [10]). They proved that, if $g(C) \geq 4$, both the gonality and the Clifford index of a curve C on a Del Pezzo surface S do not vary while moving C in its linear system, with only one class of exceptions occurring when $\deg(S) = 1$. In Section 4.3 we will show that such exceptional cases can be alternatively described as the ones where $L \otimes \omega_S$ is nef and big and the restriction of the anticanonical bundle ω_S^\vee to a general curve in $|L|$ computes its Clifford index.

In [10], the author also proved that the only curves of Clifford dimension > 1 on a Del Pezzo surface are the strict transforms of smooth plane curves and complete intersections of two cubic surfaces in \mathbb{P}^3 .

Since Green's Conjecture for smooth curves in \mathbb{P}^2 and for complete intersections of type (a, b) in \mathbb{P}^3 with $a + b = 6$ has already been verified by Loose in [11], while proving Theorem 4.1.1 we can assume that C has genus $g \geq 4$, Clifford dimension 1 and gonality $k \leq (g + 2)/2$. Under these hypotheses, we show that, if C is general in its linear system, it satisfies the linear growth condition (4.3); indeed, if $d \leq g - k + 2$, it turns out that the dimension of any irreducible component of $\mathcal{W}_d^1(|L|)$ which dominates $|L|$ under the natural projection $\pi : \mathcal{W}_d^1(|L|) \rightarrow |L|_S$ does not exceed $\dim |L| + d - k$. This follows from a parameter count for rank-2 bundles $E_{C,A}$, which are the analogue of the Lazarsfeld-Mukai bundles for K3 surfaces. The key fact proved in Section 4.4 is that, if A is a complete, base point free pencil on a general $C \in |L|$, the dimension of $\ker \mu_{0,A}$

is controlled by $H^2(S, E_{C,A} \otimes E_{C,A}^\vee)$. Hence, it is enough to find an upper bound for the number of moduli of bundles $E_{C,A}$ which are not stable; this is done in Section 4.5 by considering Harder-Narasimhan and Jordan-Hölder filtrations.

In Section 4.6 the proof of Theorem 4.1.1 is completed by showing that, under some hypotheses on L if $\deg(S) = 1$, there is an isomorphism of Koszul cohomology groups

$$K_{g-c-1,1}(S, L \otimes \omega_S) \simeq K_{g-c-1,1}(C, \omega_C),$$

which implies Green's Conjecture for *any* smooth and irreducible curve in $|L|$.

4.2 Syzygies and Koszul Cohomology

If X is a complex projective variety and L is a globally generated line bundle on it, let $S := \text{Sym}^* H^0(X, L)$ be the homogeneous coordinate ring of the projective space $\mathbb{P}(H^0(X, L)^\vee)$ and set $S(X) := \sum_m H^0(X, L^m)$. Being a finitely generate S -module, $S(X)$ admits a minimal graded free resolution

$$0 \rightarrow E_r \rightarrow \dots \rightarrow E_1 \rightarrow E_0 \rightarrow S(X) \rightarrow 0,$$

where $r = h^0(X, L) - 1$ and $E_k = \sum_{i \geq k} S(-i - 1)^{\beta_{k,i}}$. The *syzygies* of X of order k are by definition the graded components of the S -module E_k . We say that the pair (X, L) satisfies property (N_p) if $E_0 = S$ and $E_k = S(-k - 1)^{\beta_{k,k}}$ for all $1 \leq k \leq p$; in other words, the syzygies of X up to order p are linear. For instance, property (N_0) is satisfied if and only if ϕ_L embeds X as a projectively normal variety, while property (N_1) also requires that the ideal of X in $\mathbb{P}(H^0(X, L)^\vee)$ is generated by quadrics.

The most effective tool in order to compute syzygies is Koszul cohomology, whose definition is the following. Let $L \in \text{Pic}(X)$ and F be a coherent sheaf on X . The Koszul cohomology group $K_{p,q}(X, F, L)$ is defined as the cohomology at the middle-term of the complex

$$\bigwedge^{p+1} H^0(L) \otimes H^0(F \otimes L^{q-1}) \rightarrow \bigwedge^p H^0(L) \otimes H^0(F \otimes L^q) \rightarrow \bigwedge^{p-1} H^0(L) \otimes H^0(F \otimes L^{q+1}).$$

We agree that, if $F \simeq \mathcal{O}_X$, the Koszul cohomology group is simply denoted by $K_{p,q}(X, L)$. It turns out (cf. [9]) that property (N_p) for the pair (X, L) is equivalent to the vanishing

$$K_{i,q}(X, L) = 0 \text{ for all } i \leq p \text{ and } q \geq 2.$$

In particular, Green's Conjecture can be rephrased by asserting that (C, ω_C) satisfies property (N_p) whenever $p < \text{Cliff}(C)$.

In the sequel we will make use of the following results, which are due to Green. The first one is the Vanishing Theorem (cf. [9, Theorem (3.a.1)]), stating that

$$K_{p,q}(X, E, L) = 0 \text{ if } p \geq h^0(X, E \otimes L^q). \quad (4.4)$$

4 Green's Conjecture for curves on Del Pezzo surfaces

The second one (cf. [9, Theorem (3.b.1)]) relates the Koszul cohomology of X to the one of a smooth hypersurface $Y \subset X$ in the following way.

Theorem 4.2.1. *Let X be a smooth irreducible projective variety and assume $L, N \in \text{Pic}(X)$ satisfy*

$$H^0(X, N \otimes L^\vee) = 0 \quad (4.5)$$

$$H^1(X, N^q \otimes L^\vee) = 0, \forall q \geq 0. \quad (4.6)$$

Then, for every smooth integral divisor $Y \in |L|$, there exists a long exact sequence

$$\rightarrow K_{p,q}(X, L^\vee, N) \rightarrow K_{p,q}(X, N) \rightarrow K_{p,q}(Y, N \otimes \mathcal{O}_Y) \rightarrow K_{p-1,q+1}(X, L^\vee, N) \rightarrow .$$

4.3 Linear systems on Del Pezzo surfaces

A smooth irreducible surface S is called a Del Pezzo surface if its anticanonical bundle ω_S^\vee is ample. Having set $K_S := c_1(\omega_S)$, the degree of S is defined as $\deg(S) := K_S^2$. It is classically known (cf. [6]) that $1 \leq \deg(S) \leq 9$. Moreover, $\deg(S) = 9$ whenever $S = \mathbb{P}^2$, while $\deg(S) = 8$ implies that either $S = \mathbb{P}^1 \times \mathbb{P}^1$ or $S = Bl_{\mathbb{P}}(\mathbb{P}^2)$.

If $1 \leq \deg(S) \leq 7$, then S is isomorphic to \mathbb{P}^2 blown-up at $9 - \deg(S)$ points such that no three of them are collinear and no six of them lie on a conic. If S is a Del Pezzo surface of degree $s \leq 7$, then $\text{Pic}(S)$ is freely generated by the class l of the strict transform of a line and the classes e_1, \dots, e_{9-s} of the exceptional divisors, with intersection products $l^2 = 1, l \cdot e_i = 0, e_i \cdot e_j = -\delta_{ij}$. The anticanonical divisor $-K_S \sim 3l - \sum_{i=1}^{9-s} e_i$ is base point free unless $\deg(S) = 1$ (in this case it has exactly one base point) and is very ample as soon as $\deg(S) \geq 3$.

Let L be a Line bundle on S ; the Riemann-Roch Theorem gives

$$\chi(L) = 1 + \frac{c_1(L)^2 - c_1(L) \cdot K_S}{2};$$

if $L > 0$ and there exists a smooth irreducible curve $C \in |L|$ (this is always the case if L is nef and big), then

$$g(C) = 1 + \frac{c_1(L)^2 + c_1(L) \cdot K_S}{2} = \chi(S, L \otimes \omega_S) = h^0(S, L \otimes \omega_S);$$

we denote by $|L|_s$ the locus of curves inside $|L|$ which are smooth and irreducible. Notice that if L is base point free, then $h^i(S, L) = 0$ for $i \in \{1, 2\}$ (this follows by Kodaira Vanishing Theorem since $L \otimes \omega_S^\vee$ is ample).

The only (reduced and irreducible) curves with negative self-intersection on S are the (-1) -curves, that is, smooth rational curves C such that $C^2 = C \cdot K_S = -1$. If C is not a (-1) -curve, then $-K_S \cdot C \geq 2$ unless $\deg(S) = 1$ and $L \simeq \omega_S^\vee$ (cf. [10, (P6)]). For Del Pezzo surfaces, as well as for K3 surfaces, a strong version of Bertini's Theorem holds.

Theorem 4.3.1. *Let S be a Del Pezzo surface and $L \in \text{Pic}(S)$ be effective. Then, the following hold:*

- (a) *If $\deg(S) \geq 2$, then L has no base points outside its fixed components (cf. [12, Remark (0.5.3)]).*
- (b) *The line bundle L is nef if and only if it is base point free or $L \simeq \omega_S^\vee$ and $\deg(S) = 1$, and L is ample if and only if it is very ample or $L \simeq \omega_S^\vee$ and $\deg(S) \leq 2$ or $L \simeq \omega_S^{-2}$ and $\deg(S) = 1$. (cf. [7, Corollary (4.7)])*
- (c) *Assume L is nef. If $c_1(L)^2 > 0$, then a general curve in $|L|$ is smooth and irreducible. If instead $c_1(L)^2 = 0$, then $L = \mathcal{O}_S(k\Gamma)$ where $k \in \mathbb{Z}^{>0}$ and Γ is a smooth rational curve moving in a pencil (cf. [10, (P7), Proposition 3.2 (b)]).*
- (d) *If $\deg(S) < 8$, then the ampleness of L implies the nefness of $L \otimes \omega_S$ (cf. [7, Section 5])*

By Proposition 3.1 in [7], the class of the curve Γ appearing in (c) equals $l - e_i$ for some $1 \leq i \leq 9 - \deg(S)$.

The Brill-Noether theory of curves lying on Del Pezzo surfaces has been studied in details; in most of the cases, fundamental invariants, such as the Clifford index and the gonality, only depend on the linear equivalence class of the curve considered. The following result is due to Pareschi (cf. [12]).

Theorem 4.3.2. *Let S be a Del Pezzo surface of degree ≥ 2 and $C \subset S$ a smooth irreducible curve of genus g . Having set $L := \mathcal{O}_S(C)$, one has:*

- (i) *If $g(C) \geq 2$, then either the gonality of curves in $|L|_S$ is constant, or $\deg(S) = 2$ and $L \simeq \omega_S^{-2}$ (hence $g(L) = 3$).*
- (ii) *If $g(C) \geq 4$, then all curves in $|L|_S$ have the same Clifford index.*

Notice that the assumptions $g(C) \geq 2$ in (i) and $g(C) \geq 4$ in (ii) are not restricting at all because every genus 1 curve is hyperelliptic and every curve of genus at most 3 has Clifford index 0. When $\deg(S) \geq 2$, the only case of non-constant gonality is analogous to the Donagi-Morrison's example for K3 surfaces. Indeed, if S has degree 2, the anticanonical line bundle defines a double cover $\phi := \phi_{\omega_S^\vee} : S \rightarrow \mathbb{P}^2$ branched along a smooth quartic and $\omega_S^{-2} = \phi^*(\mathcal{O}_{\mathbb{P}^2}(2))$; smooth curves in the codimension-1 subspace $|\phi^*H^0(\mathbb{P}^2, (\mathcal{O}_{\mathbb{P}^2}(2)))| \subset |\omega_S^{-2}|$ are hyperelliptic because they are double covers of plane conics, while a general $C \in |\omega_S^{-2}|_S$, being isomorphic to a plane quartic, is trigonal.

In [10] Knutsen generalized the above theorem to the case $\deg(S) = 1$, showing that all curves in $|L|_S$ have the same gonality unless $L \simeq \mathcal{O}_S(-2K_S + 2E)$, where E is a (-1) -curve (thus $g(L) = 3$), or the following happens:

Example 1. *L is ample, $c_1(L) \cdot E \geq 2$ for every (-1) -curve E if $c_1(L)^2 \geq 8$, and there is an integer $k \geq 3$ such that $-c_1(L) \cdot K_S = k$, $c_1(L)^2 \geq 5k - 8 \geq 7$ and $c_1(L) \cdot \Gamma \geq k$ for every smooth rational curve such that $\Gamma^2 = 0$.*

In this case, the curves passing through the base point of ω_S^\vee form a family of codimension 1 in $|L|_S$, have gonality $k - 1$ and Clifford index $k - 3$, while a general curve

4 Green's Conjecture for curves on Del Pezzo surfaces

$C \in |L|_S$ has gonality k and Clifford index $k - 2$ (in particular, $\omega_S^\vee \otimes \mathcal{O}_C$ computes $\text{Cliff}(C)$).

Remark 4. The classes of line bundles as in Example 1 have been precisely described by Knutsen (cf. [10, p. 78]) in terms of the coefficients a and b_i such that $L \sim al - \sum a_i e_i$; one can check that $L \otimes \omega_S$ is always nef and big.

We remark that the line bundles as in Example 1 can be alternatively described as the only line bundles L such that $L \otimes \omega_S$ is nef and big and the restriction of ω_S^\vee to a general curve $C \in |L|_S$ computes its Clifford index.

Indeed, since $h^0(C, \omega_S^\vee \otimes \mathcal{O}_C) = 2$, if $\text{Cliff}(C) = \text{Cliff}(\omega_S^\vee \otimes \mathcal{O}_C)$, then C has gonality $k := -c_1(L) \cdot K_S$ and $\text{Cliff}(C) = k - 2$. The inequality $c_1(L)^2 \geq 5k - 8$ is equivalent to the trivial condition $k \leq (g + 3)/2$. Since every smooth rational curve Γ with $\Gamma^2 = 0$ moves in a base point free pencil, its restriction to C must have degree at most k . Analogously, the requirement on the intersection product with any (-1) -curve E follows from the fact that, if $c_1(L)^2 \geq 8$, the restriction of $\mathcal{O}_S(-K_S + E)$ to C defines a g_d^2 (with $d = k + c_1(L) \cdot E$) which contributes to the Clifford index.

In [10], Knutsen also proved that the Clifford index of curves in $|L|_S$ is constant except in the cases covered by Example 1. Moreover, he obtained the following:

Theorem 4.3.3. *Let S be a Del Pezzo surface and $L \in \text{Pic}(S)$ a nef line bundle such that curves in $|L|_S$ have genus $g \geq 4$. Then, all curves in $|L|_S$ have the same Clifford dimension r and, if $r \geq 2$, one of the following occurs:*

- (I) $r = 2$ and curves in $|L|_S$ are the strict transforms of smooth plane curves under a morphism $\phi : S \rightarrow \mathbb{P}^2$ which is the blow-up of \mathbb{P}^2 in $9 - \deg(S)$ points.
- (II) $r = 3$, $1 \leq \deg(S) \leq 3$ and there exist a cubic surface $S' \subset \mathbb{P}^3$ and a morphism $\phi : S \rightarrow S'$ which is the blow-up of S' in $3 - \deg(S)$ points such that curves in $|L|_S$ are the strict transforms of smooth curves in $|-3K_{S'}|$ under ϕ .

4.4 The analogue of the Lazarsfeld-Mukai bundle

Let S be a Del Pezzo surface and $C \subset S$ be a smooth, irreducible curve of genus g . If A is a complete, base point free g_d^r on C , as in the case of K3 surfaces, we consider the vector bundle $F_{C,A}$ defined by the sequence

$$0 \rightarrow F_{C,A} \rightarrow H^0(C, A) \otimes \mathcal{O}_S \xrightarrow{\text{ev}_{A,S}} A \rightarrow 0,$$

and set $E_{C,A} := F_{C,A}^\vee$. Since $N_{C|S} = \mathcal{O}_C(C)$, by dualizing the above sequence we get

$$0 \rightarrow H^0(C, A)^\vee \otimes \mathcal{O}_S \rightarrow E_{C,A} \rightarrow \mathcal{O}_C(C) \otimes A^\vee \rightarrow 0. \quad (4.7)$$

This trivially implies that:

- $\chi(S, E_{C,A} \otimes \omega_S) = h^0(S, E_{C,A} \otimes \omega_S) = g - d + r,$

4.4 The analogue of the Lazarsfeld-Mukai bundle

- $\text{rk } E_{C,A} = r + 1, c_1(E_{C,A}) = L := \mathcal{O}_S(C), c_2(E) = d,$
- $h^2(S, E_{C,A}) = 0, \chi(S, E_{C,A}) = g - d + r - c_1(L) \cdot K_S.$

As in the case of K3-surfaces, being a bundle of type $E_{C,A}$ is an open condition. Indeed, a vector bundle E of rank $r + 1$ is of type $E_{C,A}$ iff $h^1(S, E \otimes \omega_S) = h^2(S, E \otimes \omega_S) = 0$ and there exists a subspace $\Lambda \in G(r + 1, H^0(S, E))$ such that the degeneracy locus of the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \rightarrow E$ is a smooth connected curve.

Notice that the dimension of the space of global sections of $E_{C,A}$ depends not only on the type of the linear series A but also on $A \otimes \omega_S$. In particular, one has

$$\begin{aligned} h^0(S, E_{C,A}) &= r + 1 + h^0(C, \mathcal{O}_C(C) \otimes A^\vee), \\ h^1(S, E_{C,A}) &= h^0(C, A \otimes \omega_S). \end{aligned}$$

Moreover, if the line bundle $\mathcal{O}_C(C) \otimes A^\vee$ has sections, then $E_{C,A}$ is generated off its base points. In the case $r = 1$, we prove the following.

Lemma 4.4.1. *Let A be a complete, base point free g_d^1 on $C \subset S$. If either $\deg(S) \geq 2$ or $\deg(S) = 1$ and $A \not\simeq \omega_S^\vee \otimes \mathcal{O}_C$, then $h^0(C, A \otimes \omega_S) = 0$.*

Proof. Since $L \otimes \omega_S$ is effective, the short exact sequence

$$0 \rightarrow L^\vee \otimes \omega_S^\vee \rightarrow \omega_S^\vee \rightarrow \omega_S^\vee \otimes \mathcal{O}_C \rightarrow 0$$

implies that $h^0(C, \omega_S^\vee \otimes \mathcal{O}_C) \geq h^0(S, \omega_S^\vee) = 1 + \deg(S)$ and the statement follows trivially if $\deg(S) \geq 2$. Let $\deg(S) = 1$ and $h^0(C, A \otimes \omega_S) > 0$. Then it must be $h^0(C, \omega_S^\vee \otimes \mathcal{O}_C) = 2$ and $A \otimes \omega_S$ is the fixed part of the linear system of sections of A . Since A is base point free by hypothesis, then $A \simeq \omega_S^\vee \otimes \mathcal{O}_C$. \square

Under the hypotheses of the above Lemma, the bundle $E_{C,A}$ is globally generated off a finite set and $\chi(S, E_{C,A}) = h^0(S, E_{C,A}) = g - d + 1 - c_1(L) \cdot K_S$. Now, we prove an analogue of Proposition 1.4.2.

Proposition 4.4.2. *Let $C \in |L|_s$ be general and assume that either $\deg(S) \geq 2$ or $\deg(S) = 1$ and $A \not\simeq \omega_S^\vee \otimes \mathcal{O}_C$. Then for any complete, base point free pencil A on C one has:*

$$\ker \mu_{0,A} = 0 \iff H^2(S, E_{C,A} \otimes E_{C,A}^\vee) = 0.$$

Proof. The proof proceeds as in [13], hence I will not enter into details. It is easy to show that $\ker \mu_{0,A} = H^0(C, M_{C,A} \otimes \omega_C \otimes A^\vee)$, where the bundle $M_{C,A}$ is defined as the kernel of the evaluation map $ev_{A,C} : H^0(C, A) \otimes \mathcal{O}_C \rightarrow A$. Since $\det F_{C,A} = L^\vee$ and $\det M_{C,A} = A^\vee$, by adjunction one finds the following short exact sequence:

$$0 \rightarrow \omega_S \otimes \mathcal{O}_C \rightarrow F_{C,A} \otimes \omega_C \otimes A^\vee \rightarrow M_{C,A} \otimes \omega_C \otimes A^\vee \rightarrow 0. \quad (4.8)$$

The coboundary map $\delta : H^0(C, M_{C,A} \otimes \omega_C \otimes A^\vee) \rightarrow H^1(C, \omega_S \otimes \mathcal{O}_C)$ coincides, up to multiplication by a non-zero scalar factor, with the composition of the Gaussian map

$$\mu_{1,A} : \ker \mu_{0,A} \rightarrow H^0(C, \omega_C^2)$$

4 Green's Conjecture for curves on Del Pezzo surfaces

and the dual of the Kodaira spencer map

$$\rho^\vee : H^0(C, \omega_C^2) \rightarrow (T_C|L|)^\vee = H^0(C, N_{C|S})^\vee = H^1(C, \omega_S \otimes \mathcal{O}_C).$$

Indeed, as in the proof of Lemma 1 in [13], one finds a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega_S \otimes \mathcal{O}_C & \longrightarrow & F_{C,A} \otimes \omega_C \otimes A^\vee & \longrightarrow & M_{C,A} \otimes \omega_C \otimes A^\vee \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow s \\ 0 & \longrightarrow & \omega_S \otimes \mathcal{O}_C & \longrightarrow & \Omega_S^1 \otimes \omega_C & \longrightarrow & \omega_C^2 \longrightarrow 0, \end{array}$$

where the homomorphism induced by s on global sections is $\mu_{1,A}$ and the coboundary map $H^0(C, \omega_C^2) \rightarrow H^1(C, \omega_S \otimes \mathcal{O}_C)$ is (up to scalar coefficients) ρ^\vee .

If A has degree d , look at the natural projection $\pi : \mathcal{W}_d^1(|L|) \rightarrow |L|_s$. First order deformation arguments (e.g. [4, p.722]) imply that

$$\text{Im}(d\pi_{(C,A)}) \subset \text{Ann}(\text{Im}(\rho^\vee \circ \mu_{1,A})).$$

Therefore, by Sard's Lemma, if $C \in |L|_s$ is general, the short exact sequence (4.8) is exact on the global sections for any complete, base point free $A \in W_d^1(C)$, and $\ker \mu_{0,A} \simeq H^0(C, F_{C,A} \otimes \omega_C \otimes A^\vee)$.

The short exact sequence (4.7), when tensored by $F_{C,A} \otimes \omega_S$, yields

$$H^0(C, F_{C,A} \otimes \omega_C \otimes A^\vee) \simeq H^0(S, E_{C,A}^\vee \otimes E_{C,A} \otimes \omega_S)$$

because $H^i(S, F_{C,A} \otimes \omega_S) \simeq H^{2-i}(S, E_{C,A})^\vee = 0$ for $i \in \{0, 1\}$. The statement follows by Serre duality. \square

Corollary 4.4.3. *Let \mathcal{W} be an irreducible component of $\mathcal{W}_d^1(|L|)$ which dominates $|L|$ and whose general points correspond to $\mu_{L \otimes \omega_S^\vee}$ -stable bundles $E_{C,A}$; if $\deg(S) = 1$ also assume that general points of \mathcal{W} are not of the form $(C, \omega_S^\vee \otimes \mathcal{O}_C)$. Then, $\rho(g, 1, d) \geq 0$ and \mathcal{W} is reduced of dimension equal to*

$$\dim |L| + \rho(g, 1, d) = g - 1 - c_1(L) \cdot K_S + \rho(g, 1, d).$$

Proof. Let (C, A) be a general point of \mathcal{W} . If $E_{C,A}$ is stable, $E_{C,A} \otimes \omega_S$ also is. Since $\mu_{L \otimes \omega_S^\vee}(E_{C,A}) > \mu_{L \otimes \omega_S^\vee}(E_{C,A} \otimes \omega_S)$, then $H^2(S, E_{C,A}^\vee \otimes E_{C,A}) \simeq \text{Hom}(E_{C,A}, E_{C,A} \otimes \omega_S)^\vee$ vanishes. \square

4.5 Parameter count

By the analysis made in the previous section, in order to verify the linear growth condition for a general curve in $|L|_s$, it suffices to control the dimension of every dominating component $\mathcal{W} \subset \mathcal{W}_d^1(|L|)$, whose general points are pairs (C, A) such that

$A \not\simeq \omega_S^\vee \otimes \mathcal{O}_C$ and the bundle $E_{C,A}$ is not $\mu_{L \otimes \omega_S^\vee}$ -stable. Indeed, if $A \simeq \omega_S^\vee \otimes \mathcal{O}_C$ for a general point of \mathcal{W} , then $\omega_S^\vee \otimes \mathcal{O}_{C'}$ is an isolated point of $W_d^1(C')$ for every $C' \in |L|_s$.

Let A be a complete, base point free g_d^1 on a curve $C \in |L|_s$ such that the bundle $E := E_{C,A}$ is not $\mu_{L \otimes \omega_S^\vee}$ -stable and $A \not\simeq \omega_S^\vee \otimes \mathcal{O}_C$ if $\deg(S) = 1$. As in Section 3.5, by considering either the HN filtration or the JH filtration of E , we get a short exact sequence

$$0 \rightarrow M \rightarrow E \rightarrow N \otimes I_\xi \rightarrow 0, \quad (4.9)$$

where $N, M \in \text{Pic}(S)$ satisfy

$$\mu_{L \otimes \omega_S^\vee}(M) \geq g - 1 - c_1(L) \cdot K_S \geq \mu_{L \otimes \omega_S^\vee}(N), \quad (4.10)$$

and I_ξ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$ of length $l = d - c_1(N) \cdot c_1(M)$.

Lemma 4.5.1. *In the above situation, assume that C has Clifford dimension 1 and gonality $k \geq 4$ and let $k \leq d \leq g - k + 2$. Then, one of the following occurs:*

- (a) $c_1(M) \cdot c_1(N) + c_1(N) \cdot K_S \geq k - 2$;
- (b) $c_1(M) \cdot c_1(N) + c_1(M) \cdot K_S \geq k - 2$.

Proof. Being a quotient of E off a finite set, N is base component free (base point free if $\deg(S) \geq 2$) and is non-trivial since $H^2(S, N \otimes \omega_S) = 0$. As a consequence, $h^0(S, N) \geq 2$. The inequality $\mu_{L \otimes \omega_S^\vee}(M) > 0$ assures that $H^2(S, M) \simeq \text{Hom}(\mathcal{O}_S, M^\vee \otimes \omega_S)^\vee = 0$ and, if

$$\chi(S, M) = 1 + \frac{c_1(M)^2 - c_1(M) \cdot K_S}{2} \leq h^0(S, M) < 2,$$

then (4.10) implies $d \geq c_1(M) \cdot c_1(N) \geq g - 1 - c_1(L) \cdot K_S > g - 1$. Hence, we can assume both $h^0(C, N \otimes \mathcal{O}_C) \geq 2$ and $h^0(C, M \otimes \mathcal{O}_C) \geq 2$. If $h^0(S, M \otimes \omega_S) \geq 2$, then $N \otimes \mathcal{O}_C$ contributes to the Clifford index of C and (a) is satisfied since

$$\begin{aligned} k - 2 \leq \text{Cliff}(N \otimes \mathcal{O}_C) &= c_1(N) \cdot (c_1(N) + c_1(M)) - 2h^0(C, N \otimes \mathcal{O}_C) + 2 \\ &\leq c_1(N)^2 + c_1(N) \cdot c_1(M) - 2h^0(S, N) + 2 \\ &\leq c_1(N) \cdot c_1(M) + c_1(N) \cdot K_S. \end{aligned}$$

Analogously, if $h^0(S, N \otimes \omega_S) \geq 2$, then $M \otimes \mathcal{O}_C$ contributes to the Clifford index of C and we get (b). One can exclude that both $\chi(S, M \otimes \omega_S) \leq h^0(S, M \otimes \omega_S) < 2$ and $\chi(S, N \otimes \omega_S) \leq h^0(S, N \otimes \omega_S) < 2$ since this would imply

$$d \geq c_1(M) \cdot c_1(N) \geq (c_1(L)^2 + c_1(L) \cdot K_S)/2 = g - 1.$$

□

Now, having fixed a nonnegative integer l and a line bundle N such that (4.10) is satisfied for $M := L \otimes N^\vee$, we want to estimate the number of moduli of pairs (C, A) such that the bundle $E_{C,A}$ sits in a short exact sequence like (4.9). As in Section 3.5,

4 Green's Conjecture for curves on Del Pezzo surfaces

let $\mathcal{E}_{N,l}$ be the moduli stack of extensions of type (4.9) with $l(\xi) = l$, and consider the projections $p : \mathcal{E}_{N,l} \rightarrow \mathcal{M}(v(M)) \times \mathcal{M}(v(N \otimes I_\xi))$ and $q : \mathcal{E}_{N,l} \rightarrow \mathcal{M}(v(E))$.

We denote by $\tilde{P}_{N,l}$ the closure of the image of q and by $P_{N,l}$ its open substack whose \mathbb{C} -points correspond to vector bundles E satisfying $h^1(S, E \otimes \omega_S) = h^2(S, E \otimes \omega_S) = 0$ and $h^1(S, E) = 0$ (this last condition is superfluous if $\deg(S) \geq 2$ by Lemma 4.4.1). Let $\mathcal{G}_{N,l} \rightarrow P_{N,l}$ be the Grassmann bundle whose fibre over $E \in P_{N,l}(\mathbb{C})$ is $G(2, H^0(S, E))$. Having set $d := l + c_1(M) \cdot c_1(N)$, we define $\mathcal{W}_{N,l}$ to be the closure of the image of the rational map $\mathcal{G}_{N,l} \dashrightarrow \mathcal{W}_d^1(|L|)$, sending a general point $([E], \Lambda) \in \mathcal{G}_{N,l}(\mathbb{C})$ to the pair (C_Λ, A_Λ) where the evaluation map $ev_\Lambda : \Lambda \otimes \mathcal{O}_S \hookrightarrow E$ degenerates on C_Λ and has $\mathcal{O}_{C_\Lambda}(C_\Lambda) \otimes A_\Lambda^\vee$ as cokernel. The following proposition gives an upper bound for the dimension of $\mathcal{W}_{N,l}$.

Proposition 4.5.2. *Assume that general curves in $|L|_s$ have Clifford dimension 1 and gonality $k \geq 4$. Then, every irreducible component \mathcal{W} of $\mathcal{W}_d^1(|L|_s)$ which dominates $|L|$ and is contained in $\mathcal{W}_{N,l}$ satisfies:*

$$\dim \mathcal{W} \leq g - 1 - c_1(L) \cdot K_S + d - k.$$

Proof. Since N and M are line bundles, the stack $\mathcal{M}(v(M))$ has dimension -1 , while $\mathcal{M}(v(N \otimes I_\xi))$, being corepresented by the Hilbert scheme $S^{[l]}$, has dimension $2l - 1$. The fibre of p over the \mathbb{C} -point $(M, N \otimes I_\xi)$ of $\mathcal{M}(v(M)) \times \mathcal{M}(v(N \otimes I_\xi))$ is the quotient stack

$$[\mathrm{Ext}^1(N \otimes I_\xi, M) / \mathrm{Hom}(N \otimes I_\xi, M)],$$

while the fibre of q over $[E] \in \tilde{P}_{N,l}(\mathbb{C})$ is the Quot-scheme $\mathrm{Quot}_S(E, P)$, where P is the Hilbert polynomial of $N \otimes I_\xi$. The condition $\mu_{L \otimes \omega_S^\vee}(M) \geq \mu_{L \otimes \omega_S^\vee}(N)$ implies that

$$\mathrm{Ext}^2(N \otimes I_\xi, M) \simeq \mathrm{Hom}(M, N \otimes \omega_S \otimes I_\xi)^\vee = 0,$$

hence the dimension of the fibres of p is constant and equals

$$-\chi(S, N \otimes M^\vee \otimes \omega_S \otimes I_\xi) = -g + 2c_1(N) \cdot c_1(M) + c_1(M) \cdot K_S + l.$$

Moreover, the fibres of q are either all 0-dimensional or all smooth of dimension 1; this follows from the fact that $\mathrm{Hom}(M, N \otimes I_\xi) = 0$ unless $M \simeq N$ and $l = 0$, in which case $\mathrm{Ext}^1(M, N \otimes I_\xi) = H^1(S, \mathcal{O}_S) = 0$. Since every $[E] \in P_{N,l}(\mathbb{C})$ satisfies $h^0(S, E) = g - d + 1 - c_1(L) \cdot K_S$ and the fibres of $h_{N,l}$ are quotient stacks of dimension -1 (as in Section 3.5), by Lemma 4.4.1 we get:

$$\begin{aligned} \dim \mathcal{W}_{N,l} &\leq 3l - 1 - g + 2c_1(N) \cdot c_1(M) + c_1(M) \cdot K_S + 2(g - d - 1 - c_1(L) \cdot K_S) \\ &= d + g - 3 - c_1(N) \cdot c_1(M) - c_1(N) \cdot K_S - c_1(L) \cdot K_S. \end{aligned}$$

In case (a) of Lemma 4.5.1 the conclusion is straightforward. On the other hand, in case (b) we obtain

$$\dim \mathcal{W}_{N,l} \leq g - 1 + d - k - c_1(L) \cdot K_S - K_S \cdot (c_1(N) - c_1(M)),$$

and the statement follows provided that $-K_S \cdot (c_1(N) - c_1(M)) \leq 0$. Since ω_S^\vee is ample, it is enough to show that $N^\vee \otimes M$ is effective. Let $\mathcal{W} \subset \mathcal{W}_{N,l}^1$ be an irreducible component of $\mathcal{W}_d^1(|L|)$ which dominates $|L|$. By Proposition 4.4.2 we can assume that for a general $(C, A) \in \mathcal{W}$ the bundle $E_{C,A}$, which is given as an extension

$$0 \rightarrow M \rightarrow E_{C,A} \rightarrow N \otimes I_\xi \rightarrow 0, \quad (4.11)$$

satisfies $\text{Hom}(E_{C,A}, E_{C,A} \otimes \omega_C) \neq 0$. Applying $\text{Hom}(E_{C,A}, -)$ to the above short exact sequence tensored by ω_S , we get

$$0 \rightarrow \text{Hom}(E_{C,A}, M \otimes \omega_S) \rightarrow \text{Hom}(E_{C,A}, E_{C,A} \otimes \omega_S) \rightarrow \text{Hom}(E_{C,A}, N \otimes \omega_S \otimes I_\xi) \rightarrow \dots$$

By applying $\text{Hom}(-, N \otimes \omega_S \otimes I_\xi)$ (resp. $\text{Hom}(-, M \otimes \omega_S)$) to (4.11), one finds that $\text{Hom}(E_{C,A}, N \otimes \omega_S \otimes I_\xi) = 0$ (resp. $\text{Hom}(E_{C,A}, M \otimes \omega_S) \simeq \text{Hom}(N \otimes I_\xi, M \otimes \omega_S)$), hence $N^\vee \otimes M \geq N^\vee \otimes M \otimes \omega_S$ is effective. \square

4.6 Proof of Theorem 4.1.1

As remarked in the introduction, we can assume that curves in $|L|_s$ have genus $g \geq 4$, Clifford dimension 1 and (constant) gonality $4 \leq k \leq (g+2)/2$.

Having fixed $k \leq d \leq g-k+2$, Corollary 4.4.3 and Proposition 4.5.2 imply that every dominating component \mathcal{W} of $\mathcal{W}_d^1(|L|)$ has dimension $\leq \dim |L| + d - k$. As a consequence, a general curve $C \in |L|_s$ satisfies the linear growth condition (4.3), hence Green-Lazarsfeld's Gonality Conjecture holds for C and $K_{g-c-1,1}(C, \omega_C) = 0$ for $c = \text{Cliff}(C)$. If we show that the group $K_{g-c-1,1}(C, \omega_C)$ does not depend (up to isomorphisms) on the choice of C in its linear system, by semicontinuity Green's Conjecture follows for any curve in $|L|_s$ (this also provides a new proof of the constancy of the Clifford index).

Set $N := L \otimes \omega_S$; since N is nef and big, the hypotheses of Theorem 4.2.1 are satisfied. Indeed, (4.5) and (4.6) for $q = 1$ follow directly from the fact that S is regular and has geometric genus 0. Equality (4.6) for $q = 0$ is trivial since L is ample, hence $|L|$ contains a smooth, irreducible curve. For $q \geq 2$, the line bundle N^{q-1} is nef and big, hence

$$0 = H^1(S, N^{-(q-1)})^\vee \simeq H^1(S, (L \otimes \omega_S)^{q-1} \otimes \omega_S) = H^1(S, N^q \otimes L^\vee).$$

By adjunction, for any curve $C \in |L|_s$, we obtain the following long exact sequence

$$\begin{aligned} \dots &\rightarrow K_{g-c-1,1}(S, L^\vee, L \otimes \omega_S) \rightarrow K_{g-c-1,1}(S, L \otimes \omega_S) \rightarrow K_{g-c-1,1}(C, \omega_C) \\ &\rightarrow K_{g-c-2,2}(S, L^\vee, L \otimes \omega_S) \rightarrow \dots \end{aligned}$$

The group $K_{g-c-1,1}(S, L^\vee, L \otimes \omega_S)$ trivially vanishes since $H^0(S, \omega_S) = 0$. By the Vanishing Theorem (4.4) applied to $K_{g-c-2,2}(S, L^\vee, L \otimes \omega_S)$, we can conclude that

$$K_{g-c-1,1}(S, L \otimes \omega_S) \simeq K_{g-c-1,1}(C, \omega_C),$$

4 Green's Conjecture for curves on Del Pezzo surfaces

provided that $g - c - 2 \geq h^0(S, L \otimes \omega_S^2)$. We can assume $h^0(S, L \otimes \omega_S^2) \geq 2$, hence $\omega_S^\vee \otimes \mathcal{O}_C$ contributes to the Clifford index and

$$\begin{aligned} c &\leq \text{Cliff}(\omega_S^\vee \otimes \mathcal{O}_C) = -c_1(L) \cdot K_S - 2h^0(C, \omega_S^\vee \otimes \mathcal{O}_C) + 2 \\ &\leq -c_1(L) \cdot K_S - 2 \deg(S). \end{aligned} \quad (4.12)$$

Since $L \otimes \omega_S$ is nef and big, then $H^1(S, L \otimes \omega_S^2) \simeq H^1(S, L^\vee \otimes \omega_S^\vee)^\vee = 0$ and

$$\begin{aligned} h^0(S, L \otimes \omega_S^2) = \chi(S, L \otimes \omega_S^2) &= g + c_1(L) \cdot K_S + \deg(S) \\ &\leq g - c - \deg(S). \end{aligned}$$

By (4.12), we are done as, if $\deg(S) = 1$, we have $\text{Cliff}(\omega_S^\vee \otimes \mathcal{O}_C) < c$ by hypothesis.

4.7 Further remarks

Remark 5. Corollary 4.4.3 can also be proved by arguing in the following way. Let $M := M_H^{\mu_S}(\underline{c})$ be the moduli space of μ_H -stable vector bundles of fixed Chern classes $\underline{c} = (2, c_1(L), d)$ on a Del Pezzo surface S with respect to any polarization H . Since every $[E] \in M$ satisfies $\text{Ext}^2(E, E)_0 = 0$, it turns out that M is a smooth, irreducible projective variety of dimension $4d - c_1(L)^2 - 3$ (cf. [5, Remark 2.3]) as soon as it is non-empty. Let M^0 be the open subset of M parametrizing vector bundles $[E]$ such that $h^i(S, E) = h^i(S, E \otimes \omega_S) = 0$ for $i \in \{1, 2\}$ and define \mathcal{G} as the Grassmann bundle on M^0 with fibre over $[E]$ equal to $G(2, H^0(S, E))$. It's easy to see that the rational map $h : \mathcal{G} \dashrightarrow \mathcal{W}_d^1(|L|)$ is birational onto its image, that we denote by \mathcal{W} . It follows that the dimension of \mathcal{W} equals:

$$4d - c_1(L)^2 - 3 + 2(g - d - 1 - c_1(L) \cdot K_S) = 2d - 3 - c_1(L) \cdot K_S = \dim |L| + \rho(g, 1, d).$$

Remark 6. What explained in the previous remark actually holds for every anticanonical rational surface, that is, every surface S such that the anticanonical bundle ω_S^\vee is effective. Such surfaces include all blow-ups of relatively minimal models of rational surfaces, namely \mathbb{P}^2 and the Hirzebruch surfaces Σ_n for $n = 0$ and $n \geq 2$, in at most 8 points and all smooth complete toric surfaces. It seems quite likely that the techniques used in this chapter might also succeed in proving Green's Conjecture for curves C lying on some other classes of anticanonical rational surfaces, at least when the line bundle L defined by C satisfies some positive criteria.

Bibliography

- [1] M. Aprodu. On the vanishing of higher syzygies of curves. *Math. Z.*, 241(1):1–15, 2002.
- [2] M. Aprodu. Remarks on syzygies of d -gonal curves. *Math. Res. Lett.*, 12(2-3):387–400, 2005.
- [3] M. Aprodu and G. Farkas. Green’s conjecture for curves on arbitrary $K3$ surfaces. *Compos. Math.*, 147(3):839–851, 2011.
- [4] E. Arbarello, M. Cornalba, and P. A. Griffiths. *Geometry of algebraic curves. Volume II. With a contribution by Joseph Daniel Harris.*, volume 268 of *Grundle Math. Wiss.* . Berlin: Springer, 2011.
- [5] L. Costa and R. M. Miró-Roig. Rationality of moduli spaces of vector bundles on rational surfaces. *Nagoya Math. J.*, 165:43–69, 2002.
- [6] M. Demazure. Surfaces de Del Pezzo. I. II. III. IV. V. In *Seminaire sur les Singularites des Surfaces, Lect. Notes in Math.*, volume 777. Springer-Verlag, 1980.
- [7] S. Di Rocco. k -very ample line bundles on Del Pezzo surfaces. *Math. Nachr.*, 179:47–56, 1996.
- [8] S. Ehbauer. Syzygies of points in projective space and applications, Zero-dimensional schemes. In *Proceedings of the international conference held in Ravello, Italy, June 8-13, 1992.*, pages 145–170. De Gruyter, Berlin, 1994.
- [9] M. L. Green. Koszul cohomology and the geometry of projective varieties. *J. Differ. Geom.*, 19:125–171, 1984.
- [10] A. L. Knutsen. Exceptional curves on Del Pezzo surfaces. *Math. Nachr.*, 256:58–81, 2003.
- [11] F. Loose. On the graded Betti numbers of plane algebraic curves. *Manuscr. Math.*, 64(4):503–514, 1989.
- [12] G. Pareschi. Exceptional linear systems on curves on Del Pezzo surfaces. *Math. Ann.*, 291(1):17–38, 1991.
- [13] G. Pareschi. A proof of Lazarsfeld’s theorem on curves on $K3$ surfaces. *J. Algebr. Geom.*, 4(1):195–200, 1995.

Bibliography

- [14] C. Voisin. Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface. *J. Eur. Math. Soc. (JEMS)*, 4(4):363–404, 2002.
- [15] C. Voisin. Green's canonical syzygy conjecture for generic curves of odd genus. *Compos. Math.*, 141(5):1163–1190, 2005.